Essays on Inequality and Heterogeneity

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Dedication

To my family, especially my wife. It is almost impossible to think about this journey without you in it.

Abstract

Recent trends in both developed and developing economies show increasing inequality in income and wealth. Technological change is reshaping the nature of work for many, as automation, offshoring and other practices are adopted by firms around the globe. These changes to the type of jobs workers have are linked to changes in wages and labor earnings, in particular the adoption of new (worker-replacing) technologies has been linked to decreases in wages and increases in income inequality. Simultaneously, the trend towards higher inequality has sparked questions about the desirability (optimality) of inequality and whether governments should use the tools at their disposal to try to curb these trends.

My dissertation contributes to the discussion on these topics in two distinct ways. The first two chapters deal with the effects of technological change in the nature of occupations, and its effects for wage inequality, while the third chapter deals with the implications of fiscal policy (particularly capital income and wealth taxation) in the face of wealth inequality caused by differences in the rate of return across individuals. The first part of my dissertation develops a new theory of how the specific tasks carried out by workers are determined, providing a flexible framework in which to study the implications for workers of automation, offshoring, skill-biased technological change among others. I use this framework along with U.S. occupational data to study the recent adoption of automation and its effects on the wage structure. The final chapter shows how the determinants of inequality matter for determining the optimal policy in the face of inequality. In the presence of rate of return heterogeneity wealth taxes dominate capital income taxes. Relative to capital income taxes, wealth taxes benefit the individuals who are more productive, increasing the allocative efficiency in the economy, in turn leading to potentially large welfare gains despite increases in inequality.

In the first chapter, I develop an assignment model of occupations with multidimensional heterogeneity in production tasks and worker skills. Tasks are distributed continuously in the skill space, whereas workers have a discrete distribution with a finite number of types. Occupations arise as bundles of tasks optimally assigned to a type of worker. The model allows to study how occupations evolve—e.g., changes in their boundaries, wages, and employment—in response to changes in the economic environment, making it useful for analyzing the implications of automation, skill-biased technical change, offshoring, and skill upgrading by workers, among others. I characterize how the wages, the marginal product of workers, the substitutability between worker types, and the labor share depend

on the assignment. In particular, I show that these properties depend on the productivity of workers in tasks along the boundaries of their occupations.

In the second chapter I present an application of the framework developed in the first chapter. I study the rise in automation observed in recent decades. Automation is modeled as a choice of the optimal size and location of a mass of identical robots in the task space. The firm trades off the cost of the robots, which varies across the space, against the benefit of reducing the mismatch between tasks' skill requirements and workers' skills. The model rationalizes observed trends in automation and delivers implications for changes in wage inequality, unemployment, and the labor share.

Finally, the third chapter studies the quantitative implications of wealth taxation (tax on the stock of wealth) as opposed to capital income taxation (tax on the income flow from wealth) when individuals differ from each other in the rate of return they earn on their investments. With such heterogeneity, capital income and wealth taxes have opposite implications for efficiency as well as for some key distributional outcomes. Under capital income taxation, entrepreneurs who are more productive, and therefore generate more income, pay higher taxes. Under wealth taxation, on the other hand, entrepreneurs who have similar wealth levels pay similar taxes regardless of their productivity, which expands the tax base, shifts the tax burden toward unproductive entrepreneurs, and raises the savings rate of productive ones. This reallocation increases aggregate productivity and output. In the simulated model parameterized to match the U.S. data, a revenue-neutral tax reform that replaces capital income tax with a wealth tax raises average welfare by about 8 percentage points in consumption- equivalent terms. Moving on to optimal taxation, the optimal wealth tax is positive, yields larger welfare gains than the tax reform, and is preferable to optimal capital income taxes. Interestingly, the optimal wealth tax results in more equal consumption and leisure distributions (despite the wealth distribution becoming more dispersed), which is the opposite of what optimal capital income taxes imply. Consequently, wealth taxes can yield both efficiency and distributional gains.

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Chapter 1

A Task-Based Theory of Occupations with Multidimensional Heterogeneity

1.1 Introduction

Occupational labels play a useful role by summarizing the set of tasks performed by a worker. In this way, we know that dentists repair cavities, clean teeth and lecture patients about flossing, while secretaries manage schedules and send mail. Despite their usefulness, relying on occupations as descriptors of what workers do hides changes in the role of workers in production. Unlike occupational labels, the tasks actually performed by workers undergo continuous change. Dentists and secretaries perform a different collection of tasks today than they did just decades ago. Changes in the tasks performed in an occupation carry with them changes in the skills required from the worker, as well as changes in the worker's productivity and compensation.

I develop an assignment model of occupations that explicitly incorporates changes in the set of tasks involved in an occupation. In the model, production of a final good requires performing a collection of productive tasks, each generating a task-specific output. The problem is to assign workers to tasks to maximize the production of the final good. Both workers and tasks are heterogeneous along multiple dimensions as in Lindenlaub (2017). Workers differ in the skills they possess (e.g., manual, cognitive, social, etc.) and tasks differ in the skills that are involved in performing them. The relevance of multiple types of skills in determining labor market outcomes of individuals has been long recognized (Heckman and Sedlacek, 1985; Heckman, Stixrud and Urzua, 2006; Autor, Levy and Murnane, 2003;

Spitz-Oener, 2006; Black and Spitz-Oener, 2010; Deming, 2017). In particular, workers' productivity depends on the mismatch between their skills and the skills involved in the tasks they perform (i.a., Guvenen, Kuruscu, Tanaka and Wiczer, 2015b; Lise and Postel-Vinay, 2015).

To fix ideas, consider the two-dimensional setup depicted in Figure 1.1, where workers and tasks differ in cognitive and manual skills. Each point in the plane characterizes a task with a different combination of cognitive and manual skills. While some tasks are complex in terms of their cognitive skills and involve no manual skills, others involve using both types of skills. Workers are represented by points scattered in the skill space, defining a given combination of skills. Crucially, I assume that there are finitely many types of workers (e.g., x_1, x_2, x_3), while tasks are continuously distributed in the skill space.[1] Consequently, the assignment of tasks to workers for production divides the space into regions ($\mathcal{Y}_1$, $\mathcal{Y}_2$, $\mathcal{Y}_3$), resulting in bundles of tasks assigned to the same (type of) worker. The tasks in each bundle form the occupation of that worker.

The boundaries of occupations are determined by the distribution of tasks and workers across the skill space and the production technology. Figure 1.1 shows the boundaries of occupations implied by an arbitrary assignment. The shape of the boundaries under the optimal assignment depends on the technology for production of task-specific output, which determines how the mismatch between workers' skills and the skills involved in a task affects the workers' productivity. In particular, technology determines in which directions mismatch is more harmful. For example, cognitive mismatch can affect productivity more than manual mismatch, or being over-qualified can be less harmful to productivity than being under-qualified for a task. The optimal assignment seeks to maximize production by minimizing the skill mismatch, subject to the limited supply of workers of each type.

The model makes precise how the marginal productivity of workers, wages and the elasticity of substitution across workers depend on the assignment. These properties depend on the productivity of workers in tasks along the boundaries of their occupations. Boundary tasks are marginal, in the sense that they are the last tasks to be assigned to a worker. In general, they are the least productive among the tasks in the worker's occupation.[2] Wage and marginal products are thus determined by productivity at the workers' boundary tasks (i.e., how much production increases if additional tasks were reassigned to a worker). The elasticity of substitution is also determined by the boundaries, with a worker only being directly substitutable with her neighbors.

[1]This assumption is used in other task-based models of the labor market such as Rosen (1978) and Acemoglu and Autor (2011).

[2]Boundary tasks are also the most productive among tasks currently unassigned to the worker.

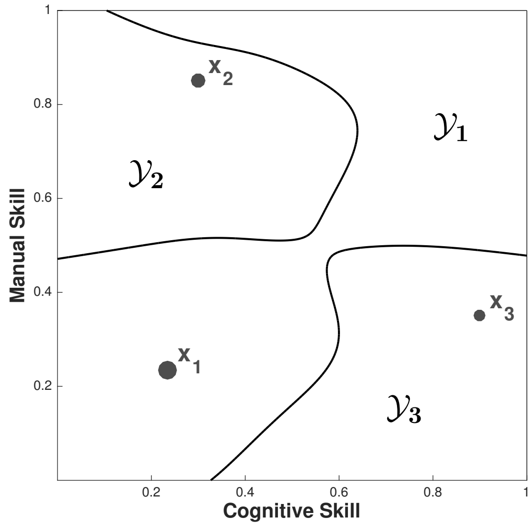


Figure 1.1: Assignment Example

Note: The figure shows an example for an assignment in a two-dimensional skill space (cognitive and manual skills). There are three types of workers $\{x_1, x_2, x_3\}$ and tasks are continuously distributed over the unit square. The assignment partitions the space into three regions $\{\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3\}$ each of which is an occupation. x_n performs the tasks in occupation $\mathcal{Y}_n$. The assignment in this figure is not necessarily optimal.

The model allows occupations to react to changes in the economic environment by changing their boundaries. In turn, changes in occupations affect the employment and wages of workers. This makes the model useful for studying the implications of automation, skill-biased technical change, offshoring, and skill upgrading by workers, among others. These changes in the economic environment manifest as changes in the production technology, the distribution of workers or the distribution of tasks. For instance, the use of information technologies (IT) and computers in the workplace make cognitive skills more relevant in production, which then affects the skill content of occupations.[3] On the other hand, the increased educational attainment of the workforce changes the distribution of skills across workers, which in turn affects the type of tasks assigned to workers of different types.

The reassignment of tasks across workers that follows a change in the environment also affects the distribution of income across workers, and between workers and firms. Who benefits from the changes in technology or the changes in the distribution of workers and tasks depends on how these changes affect the mismatch between workers and the tasks they perform. For example, the adoption of IT benefits workers with high cognitive skills over those with more manual skills. In the economy described in Figure 1.1, this increases the wage of workers of type x_3 relative to the wages of workers of type x_1 and x_2, increasing

[3]Changes in skill content of occupations go beyond the adoption of IT and the importance of cognitive skills. See Atalay, Phongthiengtham, Sotelo and Tannenbaum (2018), Rendall (2018) and Deming (2017).

inequality. The use of industrial machinery makes differences across workers in terms of their strength (a type of manual skill) less relevant for production. This makes workers x_1 and x_2 more substitutable with each other, reducing the edge that worker x_2 had from her higher manual skills. As a result wage inequality between x_1 and x_2 decreases, as does the labor share.[4]

I show how the model can be used to address other changes in the economy. I model the problem of optimal worker training as one of paying a cost to modify a worker's skill vector. This is similar to the automation problem developed in the second chapter in that the choice in both problems is a location for a worker/robot in the skill space. Unlike the automation problem, training a worker does not displace other workers, although occupations change in response to the new skill distribution. The effect on the distribution of wages is also different. Training a worker reduces the mismatch in the tasks she performs, raising her marginal product. Since wage differentials reflect differences in marginal products, training increases the differences with workers who previously earned less, and decreases the differences with workers who previously earned more than the trained worker.

I also consider how technology can change to place more weight on certain skills. This type of technological change is directed towards skills at which the workforce is already more adept, as measured by the skill-mismatch between workers and tasks'. In other words, it is optimal to specialize in skills, adapting technology to complement the skills for which the workforce is better suited, thus raising productivity. This contrasts with automation, where productivity increases by replacing workers at tasks they are not well suited for.

Finally, I extend the model by allowing tasks to be left unassigned. When not all tasks are required for the production of the final good, tasks are only performed if workers are productive enough relative to their cost (wages). This generates endogenous unemployment. I show how the value of the minimum wage affects employment and how skill accumulation by workers changes, and potentially expands, the set of tasks performed in the economy. One important consequence of allowing tasks to be left unassigned is that automation ceases to be a pure worker-replacing technology. Automation can now complement workers by taking over tasks that are either not worthwhile for workers to perform, or that are too specialized given the workers' current skills.

Related literature I adopt a task approach to production as in Rosen (1978), Autor, Levy and Murnane (2003) and Acemoglu and Autor (2011). I complement this literature by incorporating multidimensional heterogeneity in tasks and workers as in Lindenlaub (2017).

[4] As shown in Section 1.2, wages will be equalized downwards.

The main assumption I place on the model is the discreteness of the distribution of skills across workers. This assumption is motivated by the study of occupations, which arise from the assignment of bundles of tasks to a type of worker. The same assumption has been used before to address different questions. For instance, Stokey (2017) develops a model with one-dimensional heterogeneity, where a continuum of workers are assigned to finitely many tasks, to study the effects of task biased technical change on the wage structure.

Methodologically, the closest paper to mine is Feenstra and Levinsohn (1995), who use a similar setup in the context of a continuum of buyers choosing from a discrete set of products. I extend their model in a labor market context to allow for a general task-output production function and aggregation into a final good. I also provide a general proof of the existence and uniqueness of the solution using results from optimal transport theory (Villani, 2009; Galichon, 2016). This allows me to extend their differentiability results. Unlike Feenstra and Levinsohn (1995), I do not consider unobserved heterogeneity across workers and tasks. Finally, the applications to technical change, unemployment, and automation are all new.[5]

This paper is also related to the literature documenting the relevance of multiple skills in shaping labor market outcomes (Heckman and Sedlacek, 1985; Heckman, Stixrud and Urzua, 2006; Spitz-Oener, 2006; Black and Spitz-Oener, 2010; Deming, 2017). In particular, this paper is related to papers on multidimensional skill mismatch and occupational choice, and the specificity of human capital to occupations and skills, i.a., Poletaev and Robinson (2008), Kambourov and Manovskii (2009), Gathmann and Schönberg (2010), Yamaguchi (2012) Guvenen, Kuruscu, Tanaka and Wiczer (2015b), Lise and Postel-Vinay (2015) and Stinebrickner, Sullivan and Stinebrickner (2019). This literature treats the assignment of tasks to occupations as exogenous and invariant, and focuses on informational frictions. I endogenize the bundling of tasks into occupations, which depends on technology, and the demand and supply of skills.

Finally, the paper adds to the literature on the effects of automation: Acemoglu and Restrepo (2017, 2018b), Aghion, Jones and Jones (2017), Hemous and Olsen (2018), among others. In particular, I explicitly model the multidimensional nature of skill heterogeneity. This is relevant to determine the automatability of tasks as shown recently by Frey and Osborne (2017).[6] In turn, allowing for varying costs of automation across the task space lets me ask about the direction of automation. In this way, the paper provides a framework to evaluate which occupations are more likely to be affected by automation, as well as what

[5]In Feenstra and Levinsohn (1995)'s setup, the techniques I develop can be applied to the problem of designing a new product (defined by a vector of characteristics) given a distribution of consumers.

[6]Autor, Levy and Murnane (2003) also show how multiple dimensions are relevant for explaining changes in occupations. They focus on the decline of occupations intensive in routine-manual tasks.

the consequences of automation can be.

1.2 Task Assignment Model

I present a model where occupations arise as bundles of tasks assigned to workers, and the boundaries of occupations react endogenously to changes in technology (e.g., automation, skill-biased technical change) and demographics (e.g., the distribution and skills of workers). I use the model to explore how these factors change occupations, and what the effects are on worker productivity, worker compensation, and the incentives to further adopt new technologies.

The model builds on one-dimensional task-based models of production in the spirit of Rosen (1978) and Acemoglu and Autor (2011), where tasks are the basic unit of production, and tasks and workers are defined by a single dimensional measure of their 'complexity' or 'skill'. I extend the basic one-dimensional framework by incorporating multidimensional heterogeneity across workers and tasks, following Lindenlaub (2017)'s multidimensional assignment model. In the model workers are defined by a vector of skills representing their cognitive, manual, social ability, etc; tasks are defined by a vector of the skills involved in performing them. Taking into account multiple skills has been shown to be relevant when explaining educational choices (Willis and Rosen, 1979), differences in wages within demographic categories (Heckman and Sedlacek, 1985), the role of social (non-cognitive) skills relative to cognitive skills in various labor market outcomes (Heckman, Stixrud and Urzua, 2006; Deming, 2017), and the decline of occupations intensive in routine-manual tasks (Autor, Levy and Murnane, 2003).[7]

In the model, production involves the completion of a continuum of tasks by finitely many types of workers. A single type of worker can then perform various tasks; I refer to the set of tasks performed by a worker as the worker's occupation. Which tasks are assigned to each type of worker depends on the distribution of skills among workers and tasks, and on how productive workers are at different tasks. The productivity of a worker at a given task is determined in turn by how well the worker's skills match the skills used in performing the task. In what follows I describe in detail the role of workers, tasks and the production technology. Then I discuss the optimal assignment and the determinants of worker compensation.

[7]In the literature on skill formation and education, Cunha, Heckman and Schennach (2010) show that taking into account only cognitive skills can lead to wrong policy recommendations regarding investment on education.

Setup

Workers Consider an economy populated by a continuum of workers. A worker is characterized by the skills she possesses, as captured by a vector $x \in \mathcal{S} \subset \mathbb{R}^d$, where $\mathcal{S}$ is the space of skills and $d \geq 1$ is the number of skills. Vector x encodes the level of different skills the worker has, like cognitive, manual, social, etc.

There are N types of workers in the economy: $\{x_1, \ldots, x_N\} \equiv \mathcal{X}$. x_n is the skill vector of workers of type n. There is a mass p_n of workers of type x_n, so that the total mass of workers in the economy is $P = \sum_{n=1}^{N} p_n$. Each worker is endowed with one unit of time. This implies that workers of type n have a total of p_n units of time available to work. Workers can either work or be unemployed. If unemployed, a worker receives a payment $\underline{w} \geq 0$. Workers supply their time inelastically at any wage $w \geq \underline{w}$.

Tasks There is a single final good produced in the economy that aggregates the output of all workers across productive tasks. In particular, production of the final good involves completing a continuum of differentiated tasks. Let $\mathcal{Y} \subseteq \mathcal{S}$ denote the set of tasks used in production. Tasks $y \in \mathcal{Y}$ differ in the skills involved in performing them, and how many times they must be performed. One unit of time is required to perform a task once. To make this precise, I represent a task y by a vector of skills, so that $y \in \mathcal{Y} \subseteq \mathcal{S}$, and denote the density of tasks used in production by $g : \mathcal{Y} \to \mathbb{R}_+$. I assume throughout that:

i $g : \mathcal{Y} \to \mathbb{R}_+$ is an absolutely continuous (a.c.) function with an associated a.c. measure G on $\mathcal{Y}$;

ii there are enough workers to complete all tasks, i.e. $G(\mathcal{Y}) = \int_{\mathcal{Y}} g(y)\, dy \leq P$;

iii the set of tasks $\mathcal{Y}$ is compact.

Task-output Workers vary in their productivity across tasks depending on the match between the skills they possess (x) and the skills involved in performing the task (y). q describes how productive a worker with skills x is when performing task y. These differences play a crucial role in determining the assignment of tasks to workers and through it the overall productivity of each worker type and the substitutability across workers. As will be discussed later in this Section, the optimal assignment will balance the desire to minimize the mismatch between workers and the tasks they perform, with the capacity constraints imposed by the limited availability of workers.

I will denote by $q(x, y)$ the worker/task-specific output generated by a worker of type x performing task y. If a task is not assigned to any worker, then no output is generated for that task (abusing notation: $q(\emptyset, y) = 0$ for all $y \in \mathcal{Y}$). Further properties of the task-output function $q : \mathcal{X} \times \mathcal{Y} \rightarrow \mathbb{R}_+$ will be specified later.

Assignment As mentioned above, production of the final good combines output from all workers. The output of a worker depends in turn on which task she performs, according to the productivity of the worker implied by q. Because of this, the production of the final good will depend on how tasks are assigned to workers. The assignment of tasks to workers is described by a function $T : \mathcal{Y} \rightarrow \mathcal{X}$, so that task y is performed by worker $T(y) \in \mathcal{X}$.

Many tasks can be assigned to the same worker type. I collectively refer to the set of tasks performed by a type of worker as the occupation of the worker. The assignment T determines which tasks are bundled into the occupation of each worker. The occupation of workers of type x_n is:

$$\mathcal{Y}_n = T^{-1}(x_n) = \{y \in \mathcal{Y} \mid x_n = T(y)\} \tag{1.1}$$

Occupations form a partition of the space of tasks into at most N cells.[8] Figure 1.1 shows an example of an assignment that partitions the space of tasks into three occupations, corresponding to three worker types.

An assignment is deemed *feasible* if workers have enough time to supply all the time demanded by their occupation. This time is given by the number of tasks in the worker's occupation. The demand for worker n's time is:

$$D_n = \int_{\mathcal{Y}_n} dG \tag{1.2}$$

An assignment is feasible if $D_n \leq p_n$ for all $n \in \{1, \ldots, N\}$.

Remark. The definition of the assignment function implicitly assumes that all tasks are assigned to a worker in $\mathcal{X}$. This is without loss given the way in which output from all worker/task pairs is aggregated into the final good (see equation 1.3 below). I will expand on this in the next subsection where I also discuss how to explicitly include the possibility of not performing some of the tasks.

Final good production The production of a final good aggregates the output from all worker/task pairs through a Cobb-Douglas technology.[9] Given an assignment T, total

[8] It is possible that $\mathcal{Y}_n = \emptyset$, so that no task is assigned to worker x_n.

[9] The aggregator does not need to be of the Cobb-Douglas type. Results hold for aggregators of the CES family: $F(T) = \left(\int (q(T(y), y))^{\frac{\sigma-1}{\sigma}} dG(y)\right)^{\frac{\sigma}{\sigma-1}}$, with $\sigma \geq 1$. See Appendix A.2.

output is:

$$F(T) = \exp\left(\int_{\mathcal{Y}} \ln q\left(T(y), y\right) dG\right) \tag{1.3}$$

Under this technology, production of the final good only takes place if all tasks are assigned and performed. Recall that if a task is left unassigned $q(\emptyset, y) = 0$. In this sense, technology resembles a continuous version of Kremer (1993)'s O-Ring production function. In order to make the comparison precise, it is necessary to change the interpretation of q. Consider a continuous production line indexed by $y \in \mathcal{Y}$, at each point in the production line a fatal error can occur that terminates the production process in failure. The arrival rate of an error is given by $\ln q(x, y) \geq 0$ and depends on the point in the production process (y) and the worker assigned to that point (x). The probability that no error arrives at the end of the whole process is given by (1.3). Thus, $F(T)$ can be interpreted as expected output given an assignment T. See Sobel (1992) for another application of this idea.

Optimal assignment

The problem is to find a feasible assignment that maximizes output:

$$\max_{T} F(T) \qquad \text{s.t. } \forall_n \, D_n \leq p_n \tag{1.4}$$

The assignment determines how tasks are divided into occupations. The exact form of the assignment depends on three factors. First, the distribution of skills in the workforce, which is described by the skill vectors of N types of workers (x_n), and the mass of workers of each type (p_n). Second, the distribution of tasks involved in production, captured by the function g. Finally, the production technology embedded in q, which determines how workers and tasks' characteristics interact in production. Changes in any of these factors translate into changes to the optimal assignment of tasks to workers, thus affecting the boundaries of occupations and the production of the final good.

Even though the optimal assignment cannot be fully characterized without completely specifying the environment, it is possible to guarantee the existence and uniqueness of a solution by imposing conditions only on the production technology q. The following proposition makes this precise:

Proposition 1. *Consider the optimal assignment problem in (1.4).*
If q is such that:

1. *Every worker/task pair produces positive output: $q(x, y) > 0$ for all pairs $(x, y) \in \mathcal{X} \times \mathcal{Y}$.*

2. *$q(x,\cdot)$ is upper-semicontinuous in y given $x \in \mathcal{X}$.*
3. *q discriminates across workers in almost all tasks: for all $x_n \neq x_\ell$, $q(x_n, y) \neq q(x_\ell, y)$ G-a.e.*

Then there exists a $(G-)$unique solution $T^\star$ to the problem in (1.4). Moreover, there exist a unique $\lambda^\star \in \mathbb{R}^N$ with $\min \lambda_n^\star = 0$ such that $T^\star$ is characterized as:

$$T^\star(y) = argmax_{x\in\mathcal{X}} \left\{ \ln q(x,y) - \lambda^\star_{n(x)} \right\} \tag{1.5}$$

where $n(x)$ gives the index of a type of worker $x \in \mathcal{X}$.

Proof. The result is established by re-expressing the problem in (1.4) as an optimal transport problem. The proof is divided into three Lemmas that relax the problem in (1.4) by allowing non-deterministic assignments, and then construct a solution by means of the dual of the relaxed problem. The solution is shown to be unique and to characterize a deterministic assignment $T^\star$ according to equation 1.5. The Lemmas follow from applying Theorems 5.10 and 5.30 in Villani (2009) summarized in Theorem 1 of Appendix A.1. All Lemmas are stated and proven in Appendix A.2.

□

The first two conditions on q in Proposition 1 are technical and ensure that the theory of duality applies to the problem. The value of $\lambda^\star$ is obtained from the solution to the dual problem to A.2. The third condition plays a crucial in role in establishing the existence and uniqueness of an optimal assignment function $T^\star$. The condition makes it possible to distinguish between workers in each task by demanding injectivity of q in x given y. It plays the same role as the 'twist condition' of Carlier (2003), the condition for positive assortative matching in Lindenlaub (2017), and the single-dimensional Spence-Mirrlees single-crossing property. However, the injectivity condition I assume is less restrictive than the 'twist condition' since it does not involve differentiability of q, moreover, it is simpler to verify in practice since there are finitely many types of workers.

The characterization of the optimal assignment in 1.5 allows me to solve the problem in a task-by-task basis, and give a more explicit characterization of the occupations in terms of the production technology q:

$$\mathcal{Y}_n = \{ y \in \mathcal{Y} \,|\, \forall_\ell \ \ln q(x_n, y) - \lambda_n^\star \geq \ln q(x_\ell, y) - \lambda_\ell^\star \} \tag{1.6}$$

Tasks are optimally assigned to workers that are more productive at performing them. That is, workers with lower skill mismatch. The role of the multiplier $\lambda^\star$ is to penalize the output of a worker in a given task to balance the demand for that type of worker with the limited supply of hours (p_n). The boundaries of an occupation are formed by task $y \in \partial\mathcal{Y}_n$ for which the inequality in (1.6) is met with equality for some k.

Even though it is possible not to perform a task, by leaving it unassigned, this does not happen under the optimal assignment. It is optimal to assign all tasks because there is no production of the final good if one task is left unassigned (recall that $q(\emptyset, y) = 0$ for all y). In Section 1.4, I consider an alternative interpretation of the production technology under which tasks can be left unassigned.[10] Doing so gives rise to endogenous unemployment in the model. Unemployment depends on how many (and which) tasks are not performed in the optimal assignment. The results in Proposition 1 do not change by considering the possibility of leaving tasks unassigned, but worker's compensation does change. I expand on this in the next subsection and in Section 1.4.

Indirect production function

The production technology described above depends not only on how many workers of each type are used but also on which tasks are performed by each of them; unlike the 'canonical' production function where the roles of each input (in this case each type of worker) are predetermined and unchanging. In the model described above the amount of an input (a type of worker) used in production and what that input is used for are not the same (Autor, 2013). As a consequence, the relation between inputs and output depends on how the tasks are assigned to workers, and how the assignment itself changes as the inputs vary.

The aggregate role of workers in production is captured by the value of the assignment problem (1.4). The value of the problem defines an indirect production function that depends on the availability of workers in the economy:

$$V(p_1, \ldots, p_N) = \max_{T} F(T) \qquad \text{s.t. } \forall_n \, D_n \leq p_n \tag{1.7}$$

Function V describes how production changes when the composition of the workforce changes, allowing for workers to be re-assigned optimally across tasks.

The properties of workers in production, such as their marginal product and the substitutability across different types of workers, are determined by how the assignment reacts to changes in the supply (distribution) of workers. In particular, the properties of workers in production depend on their productivity along the boundaries of occupations, and on how those boundaries react to changes in the environment.

[10] When a task is left unassigned it is taken out of the mix of tasks instead of having output be zero. This amounts to changing the set over which the integral in (1.3) is taken.

Marginal products and worker compensation

The marginal product of workers of type n is obtained from V as the change in output if the supply of worker's of type n (p_n) were to increase.[11] The marginal product is given by the percentage increase in output obtained from adding more workers of the given type. This is made clear by relating the marginal product to the solution of the dual problem (A.3):

$$\mathrm{MP}_n = \frac{\partial V\left(p_1, \ldots, p_N\right)}{\partial p_n} = F\left(T^{\star}\right) \lambda_n^{\star} \tag{1.8}$$

The result follows from the envelope theorem (Milgrom and Segal, 2002) and is proven in Lemma 4 in Appendix A.2.

To see how the value of $\lambda^{\star}$ relates to the productivity of each type of worker, we must first determine how the assignment responds to an increase in the supply of workers. When the supply of workers of type n increases, the additional workers are only used if tasks are re-assigned to them from other workers. The first tasks to be reassigned are those in the boundaries of occupations. Consider the occupations of two types of workers, n and ℓ, all tasks in the boundary of the occupations, i.e. $y \in \mathcal{Y}_n \cap \mathcal{Y}_{\ell}$, satisfy:

$$\lambda_n^{\star} - \lambda_{\ell}^{\star} = \ln q\left(x_n, y\right) - \ln q\left(x_{\ell}, y\right) \tag{1.9}$$

Then the difference in the multipliers $\lambda_n^{\star}$ and $\lambda_{\ell}^{\star}$ is given by the log difference in task output along the boundary between workers n and ℓ. That is, the percentage increase (or decrease) in output if the tasks along the boundary are re-assigned from ℓ to n.[12] It is only optimal to make use of the additional supply of workers if output increases along the boundary of worker's n occupation ($\partial \mathcal{Y}_n$).

If tasks are reassigned to the additional type n workers, workers along the boundaries of $\mathcal{Y}_n$ are displaced. This process generates an excess supply of workers of other types, giving rise to a new round of re-assignment along the boundaries of these workers. Following the process reveals an ordering of workers by productivity, with the least productive worker being displaced by increases in the supply of more productive workers. As a consequence,

[11] This definition of marginal product takes into account how the assignment changes optimally in response to the increase in the supply of workers of type n. It is also possible to define an arbitrary measure for the marginal product of a type n worker at a given task y, given some arbitrary assignment T. I discuss it in Appendix A.3.

[12] It is useful to consider an example with finitely many tasks, say $\{y_1, y_2\}$. Then total output is given by $F\left(T\right) = q_1\left(x_n\right) q_2\left(x_{\ell}\right)$. If the assignment changes by having worker x_n perform both tasks the new output is $F\left(T^{'}\right) = \frac{q_2(x_n)}{q_2(x_{\ell})} F\left(T\right)$. Then $\ln \frac{F\left(T^{'}\right)}{F(T)} = \ln \frac{q_2(x_n)}{q_2(x_{\ell})} = \lambda_n - \lambda_{\ell}$, so that output increases by $100\left(\lambda_n - \lambda_{\ell}\right)\%$.

the least productive worker has zero marginal product.[13] Increases in the supply of that type of workers do not increase output because the additional workers are left unassigned (unemployed).

The total gain in output from the initial increase in the supply of workers of type n takes into account the increase in output from all the re-assignments. Using the relation in (1.9), and recalling that $\min \lambda_k = 0$, we get a total increase in output of λ_n as in (1.8).

The value of the marginal product affects how workers are compensated. To see this consider how the optimal assignment of tasks to workers can be implemented by a price-taking firm seeking to maximize profits. The firm's problem is:

$$\max_{T} F(T) - \sum_{n=1}^{N} w_n D_n(T)$$

where w_n is the wage paid to a worker of type n, and D_n is the demand for workers of type n, given by (1.2). This problem is equivalent to the optimal assignment problem in (1.4) if the wages correspond to the multipliers of the feasibility constraint of each worker type. This is the case if wages are of the form:

$$w_n = F(T^\star)\lambda_n^\star + \kappa \qquad \text{where } \kappa \geq \underline{w} \tag{1.10}$$

The wages that decentralize the optimal assignment are given by the marginal product of each worker under the optimal assignment, plus a constant that guarantees that all workers receive at least their outside option. The level of the wages is not pinned down in the problem because only the difference in wages affects the assignment (see equation 1.6). Recall that in order to produce the firm has to employ a total of $G(\mathcal{Y})$ hours, independently of which workers are hired. So, if all wages increase by κ the total wage bill increases by $\kappa G(\mathcal{Y})$ regardless of the assignment. From the point of view of the firm the constant κ acts as a fixed cost, and thus, it has no effect on the assignment.[14]

An alternative to determining the level of wages in the economy is to allow for tasks to

[13]The property that the least productive worker has zero marginal product is induced by the capacity constraint on the set of tasks and the times each task can be performed. It is also what motivates the normalization of the multiplier λ in proposition 1.

[14]The indeterminacy of the level of worker compensation is a common feature of assignment models (Sattinger, 1993). Only total surplus and differences across workers are pinned down by the optimality conditions. This result is not a feature of the discreteness in the distribution of workers, see Lindenlaub (2017). The level of worker compensation depends on additional assumptions. For example, having excess workers ($P > G(\mathcal{Y})$) implies that (at least) some type of worker will be partially unassigned (unemployed), driving down the wage for that type of worker to $\underline{w}$. This will be the case when I introduce automation in Section 2.2.1. Once the wage of one type of worker is known the other wages are implied by differences in their marginal product (see equation 1.9).

remain unassigned as presented in Section 1.4, or because of the introduction of automation or offshoring as presented in Section 2.2.1. The threat of leaving a task unassigned lowers the wage of the least productive worker to its outside option, effectively pinning down $\kappa = \underline{w}$ as the worker with the lowest productivity has zero marginal product. Which tasks are performed under the optimal assignment is then a function of $\underline{w}$. A higher outside option for workers makes unprofitable to perform more tasks and can induce unemployment among workers of different types.

Substitutability across workers

The substitutability of different types of workers in production plays an important role in policy analysis, and in determining the effects of changes to technology and the distribution of skill supply and demand. The substitutability is usually measured by the elasticity of substitution, which, under the 'canonical' production function is typically assumed to be constant and invariant to changes in technology. This is the case when labor is combined through a CES aggregator as in Katz and Murphy (1992) (see Acemoglu and Autor (2011) for more references). Yet, the substitutability between different types of workers depends on the characteristics of the assignment, and reacts endogenously to changes in technology and the distribution of workers and tasks.

Explicitly modeling the assignment of tasks to workers makes it possible to measure how substitutable workers are depending on which tasks they perform. This includes how substitutable are 'low' and 'high' skilled workers, or workers specialized in cognitive vs manual skills. Intuitively, workers performing similar tasks are more substitutable, as are workers with similar skills. In order to make these results precise, I compute the elasticity of substitution under the optimal assignment.

Since there are in general more than two types of workers the appropriate measure of substitutability is given by the Morishima elasticity of substitution (Blackorby and Russell, 1981, 1989).[15] The elasticity of substitution between workers of type n and ℓ is:

$$M_{\ell n} = \mathcal{E}_{\ell n} - \mathcal{E}_{nn} \tag{1.11}$$

where $\mathcal{E}_{\ell n} = \frac{\mathrm{MP}_n}{D_\ell} \frac{\partial D_\ell}{\partial \mathrm{MP}_n}$ is the cross elasticity of demand for worker k with respect to a change in the marginal product of worker n.[16] Changes in the marginal product of worker

[15] See Baqaee and Farhi (2018) for a recent application of the Morishima elasticity in an input-output network setting.

[16] The Morishima elasticity of substitution measures the effect on the ratio of optimal demands for two inputs (in this case two types of workers, ${D_\ell}/{D_n}$) given by a (proportional) change of the ratio of marginal products (${\mathrm{MP}_n}/{\mathrm{MP}_\ell}$). Recall that marginal products and wages move together. When there are more than

x_n are captured by changes in $\lambda_n^\star$ (see equation 1.8). Knowing this, it becomes clear from the characterization of occupations in (1.6) that $\mathcal{E}_{nn} < 0$ and $\mathcal{E}_{n\ell} \geq 0$. That is, increasing $\lambda_n^\star$ decreases the demand for worker x_n and (weakly) increases the demand for other workers. From the point of view of worker compensation, increasing $\lambda_n^\star$ raises the cost of worker n, causing the firm to substitute it for other workers. Because of this the elasticity of substitution is always positive in the model.[17]

Yet, the relevant measure for direct substitutability between workers is the cross-elasticity $\mathcal{E}_{n\ell}$. In a setting with more than two inputs, the ratio ${}^{D_\ell}/_{D_n}$ can change in response to changes in the marginal product of x_n without the demand for worker ℓ being affected. Because of this, the elasticity of substitution between workers n and ℓ is at least equal to $\mathcal{E}_{nn}$, being only greater if there is direct substitution between the two workers, that is, if the demand for worker ℓ changes when the marginal product of n changes. As shown in proposition 2 this happens only if workers n and ℓ share a boundary.

To obtain the magnitude of the elasticity of substitution between two workers it is necessary to determine how much their demands change with the value of $\lambda^\star$. Looking again at the characterization of occupation in (1.6) the change in the demand will depend on how sensitive the boundaries of the occupation are to changes in $\lambda_n^\star$. The sensitivity of the boundaries depends in turn on the slope of the production function q evaluated at the boundary tasks, see (1.9). Specifying a functional form on q becomes necessary to completely characterize $\mathcal{E}_{nn}$ and $\mathcal{E}_{kn}$.

Linear-Quadratic task-output production function

I now introduce a specific functional form for task-output. I follow Tinbergen (1956), Feenstra and Levinsohn (1995) and Lindenlaub (2017) in assuming a linear-quadratic production function:

$$q(x,y) = \exp\left(a_x^{'}x + a_y^{'}y - (x-y)^{'} A (x-y)\right) \tag{1.12}$$

The output from all worker/task pairs is combined into a final good according to the Cobb-Douglas aggregator in (1.3).

Under (1.12) the productivity of a worker at a given task depends on the skill mismatch

two workers the direction of the change in the ratio of marginal products matters since the demands for inputs changes differently if MP_n or MP_ℓ vary, see Blackorby and Russell (1989, pg 885). Because of this the elasticity is in general asymmetric. I consider a change in the ratio of marginal products in the direction of MP_n:

$$M_{\ell n} = \frac{\partial \ln {}^{D_n}/_{D_\ell}}{\partial \ln \text{MP}_n} = \frac{\text{MP}_n}{D_\ell}\frac{\partial D_\ell}{\partial \text{MP}_n} - \frac{\text{MP}_n}{D_n}\frac{\partial D_n}{\partial \text{MP}_n} = \mathcal{E}_{\ell n} - \mathcal{E}_{nn}$$

[17] Workers satisfy the Kelso and Crawford (1982) gross substitutes condition.

between the worker's skills (x) and the skills involved in performing the task (y), measured by the weighted distance between worker and task's skills.[18] Matrix A controls the weights of each skill in the mismatch; it is assumed to be symmetric and positive definite. The higher the weight of a skill the more important it is for production; mismatch in that skill hurts production more. The linear terms ($a_x^{'}x$ and $a_y^{'}y$) capture more skilled workers having an absolute advantage in production, and the value of output generated by tasks involving higher skill levels being higher.

The functional form in (1.12) greatly simplifies the characterization of occupations in the optimal assignment.[19] In particular, boundaries take the form of hyperplanes whose normal vectors depend on matrix A and the difference in skills between neighboring workers. This is made clear by replacing (1.12) in condition (1.9). The boundary between the occupations of workers x_n and x_ℓ is:

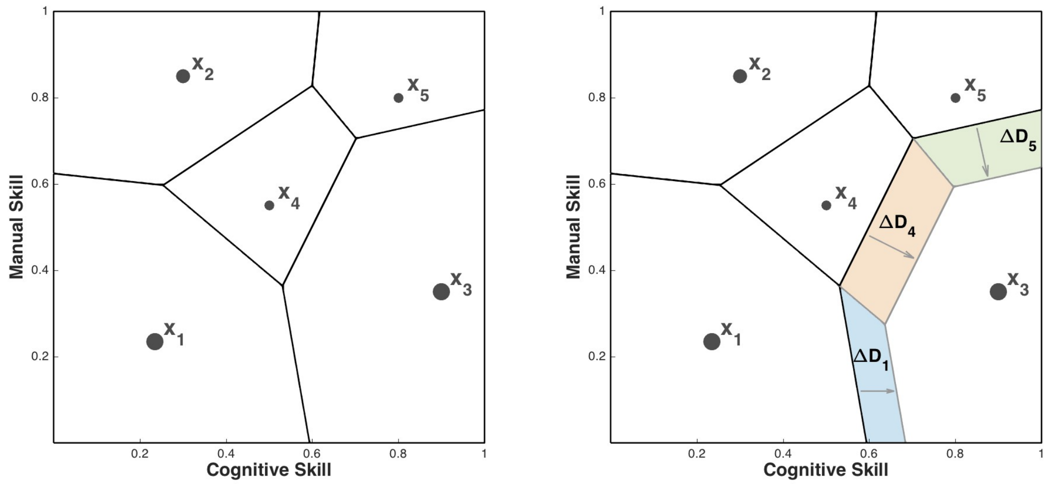
$$y \in \mathcal{Y}_n \cap \mathcal{Y}_\ell \longleftrightarrow 0 = y^{'} \underbrace{A\left(x_\ell - x_n\right)}_{\text{Normal Vector}} - \frac{1}{2} \underbrace{\left(x_\ell^{'} A x_\ell - x_n^{'} A x_n + a_x^{'}\left(x_\ell - x_n\right) + \lambda_\ell^\star - \lambda_n^\star\right)}_{\text{Intercept}} \quad (1.13)$$

Figure 1.2a shows the form of the optimal assignment when q is given by (1.12). A worker will perform the tasks closest to her skills, for which she has the least mismatch, conditioned on the limited supply of workers (feasibility constraint in 1.4). The location of the boundaries depends on the value of the multipliers $\lambda^\star$, but the slope depends on the relationship between workers' skills and the production technology embodied by A. Figure 1.2b illustrates this by increasing the value of $\lambda_3^\star$. When $\lambda_n^\star$ increases the boundaries of the occupation of worker n will shift 'inward' in a parallel fashion, reducing the demand for x_n and increasing the demand for all its neighbors. If an occupation $\mathcal{Y}_m$ does not share a border with $\mathcal{Y}_n$, it is not directly affected by changes in $\lambda_n^\star$ (see the boundaries of $\mathcal{Y}_2$ in Figure 1.2b).

The geometric structure induced by adopting the functional form in (1.12) makes it possible to characterize the change in demand following a change in $\lambda^\star$ in a general way (Feenstra and Levinsohn, 1995). The change in demand is always given by the area of a (hyper)trapezoid,

[18]The dependance of production on the mismatch between worker and task skills is similar in spirit to Lazear (2009)'s skill weights approach, where he studies job-specific skills, and to the skill mismatch studies of Guvenen et al. (2015b) and Lise and Postel-Vinay (2015), who study earnings differential across occupations and the accumulation of skills by workers.

[19]Under (1.12) there is an equivalence between the optimal assignment and the partition induced by a power diagram. A power diagram partitions a space into cells that minimize the power between a node (x) associated with the cell and the points y in the cell. The outcome is a partition of the space into convex polyhedra defined by hyperplanes. The power function between two points is $\text{pow}\left(x, y\right) = d\left(x, y\right)^2 - \mu$, where $d\left(x, y\right)$ is a distance and $\mu \in \mathbb{R}$. This relation is noted by Galichon (2016, ch. 5) and is treated formally by Aurenhammer et al. (1998).



(a) Optimal assignment

(b) Changes in demand when $\lambda_3^\star$ increases

Figure 1.2: Assignment Example - Quadratic Mismatch Loss

Note: The figures shows the assignment in a two-dimensional skill space (cognitive and manual skills). Five types of workers are considered $\{x_1, \ldots, x_5\}$ with mass $P = \{0.3, 0.2, 0.3, 0.1, 0.1\}$. Tasks are uniformly distributed over the unit square, i.e $\mathcal{Y} = [0,1]^2$ and $g(y) = 1$ for all y. The production function q is given by (1.12) with $A = I_2$, the value of a_x and a_y does not change the optimal assignment.

formed as the plane that defines the boundary between occupations moves (see Figure 1.2b). I exploit the geometric structure of the problem to compute closed-form expressions for the derivates of demand (Proposition 2). The cross derivative of demand depends on how exposed two workers are to one-another, measured by the length of their boundary, and how similar their skills are, measured by the weighted distance between their skill vectors (x_n and x_ℓ). When the demand for a worker changes, it is optimal to make the adjustment along the boundaries. Because of this, workers with longer boundaries are more substitutable; moreover, workers are only directly substitutable with their occupational neighbors. How much the boundary reacts to a change in demand depends on how similar workers are at performing tasks. The closer the skills of the workers are, the more substitutable along their boundary. Finally, the second part of the proposition follows from noting that the set of tasks is fixed, so the total effect of the change in demand as λ_n changes must be zero.

Proposition 2. *Let $\lambda \in \mathbb{R}^N$ be a vector of multipliers. If q is continuous then D_n is continuously differentiable with respect to λ and:*

i $\forall_{\ell \neq n} \quad \frac{\partial D_n}{\partial \lambda_\ell} = \frac{area(\mathcal{Y}_n \cap \mathcal{Y}_\ell)}{2\sqrt{(x_n - x_\ell)' A' A (x_n - x_\ell)}} = \frac{\int_{\mathcal{Y}_n \cap \mathcal{Y}_\ell} dG}{2\sqrt{(x_n - x_\ell)' A' A (x_n - x_\ell)}} \geq 0$

ii $\frac{\partial D_n}{\partial \lambda_n} = -\sum_{\ell \neq n} \frac{\partial D_\ell}{\partial \lambda_n} < 0$

The proof of Proposition 2 is presented in Appendix A.2, it extends the results of Feenstra

and Levinsohn (1995) by applying Reynolds' transport theorem (see Theorem 2 in Appendix A.1) to compute the change in demand for arbitrary configurations of workers (x).
Using part two of Proposition 2 the expression for the Morishima elasticity becomes:

$$M_{\ell n} = \left(1 + \frac{D_\ell}{D_n}\right)\mathcal{E}_{\ell n} + \sum_{m \neq n,\ell} \frac{D_m}{D_n}\mathcal{E}_{mn} \tag{1.14}$$

The elasticity of substitution between workers x_n and x_ℓ is a weighted average of the cross-elasticities of demand of all workers, with the weights given by the demand of each type of worker relative to worker n's demand. The elasticity includes the direct substitution effect between n and ℓ, and the secondary effects induced by the substitution of worker n for other workers ($m \neq n, \ell$). When two workers do not share a boundary ($\mathcal{Y}_n \cap \mathcal{Y}_\ell = \emptyset$) the direct effect disappears since the cross-demand elasticity is zero, but the elasticity of substitution is not zero because it takes into account the changes in the assignment through the boundaries of $\mathcal{Y}_n$. When there are only two types of workers the second term vanishes in (1.14), and, noting that $\frac{\partial D_n}{\partial \lambda_n} = -\frac{\partial D_\ell}{\partial \lambda_n} = \frac{\partial D_\ell}{\partial \lambda_\ell} = -\frac{\partial D_n}{\partial \lambda_n}$, we get symmetry.
Adopting the functional form in (1.12) also makes it possible to characterize the differences in marginal products across workers in terms of the differences in skills and skill mismatch. Manipulating equation (1.13) and recalling that min $\lambda_n = 0$ it is possible to get the following expression for λ_n:

$$\lambda_n = \underbrace{a_{x'}\left(x_n - \underline{x}\right)}_{\text{Difference in Skills}} \underbrace{- \underbrace{\left(x_n - y_n\right)' A \left(x_n - y_n\right)}_{x_n \text{ mismatch at boundary}} - \underbrace{\left(\underline{x} - \underline{y}\right)' A \left(\underline{x} - \underline{y}\right)}_{\underline{x} \text{ mismatch at boundary}}}_{\text{Difference in Mismatch}} \tag{1.15}$$

where $\underline{x}$ are the skills of the lowest paid worker (the worker with $\lambda_m = 0$), and y_n and $\underline{y}$ are boundary tasks of workers x_n and $\underline{x}$ respectively.
The marginal product of a worker is defined relative to the marginal product of the lest productive worker. This follows from the ordering of workers discussed earlier. Assigning additional tasks to workers of type x_n implies taking tasks away from workers of type $\underline{x}$. Under (1.12) the marginal product of a worker depends on skill level, relative to those of the least productive worker, and on the mismatch at the boundary tasks.

1.3 Directed technical change and skill accumulation

Changes in technology are a major factor in shaping the way in which tasks are assigned to workers. For instance, the increase of information technology (IT) in the workplace has shifted focus from manual to cognitive skills, and changed the distribution of tasks across occupations (e.g., clerical and secretarial jobs). More directly, automation technologies and offshoring have replaced workers in performing certain tasks across manufacturing jobs, customer services, and accounting among others.

I consider here a particular form of technical change and study how it affects the division of tasks into occupations. This form of technical change is captured in innovation in skill-enhancing technology, such as IT in the modern workplace, or the power loom in the 18th and 19th centuries. This innovation changes the productivity of workers across tasks, inducing a reassignment of tasks to reduce mismatch across occupations. Technical change is followed by changes in the role of workers in production, affecting their productivity and substitution patterns.

This type of technical change can be directed towards specific skills with the aim of increasing production. Production is increased the most by reducing the mismatch in between tasks and workers, by increasing the weight on skills for which the workforce is better suited.

The answer to which skills become more important for production, depends on the joint distribution of skills requirements across tasks and skill endowments of workers, and its interaction within the production technology. Moreover, the answer depends on how changes in technology influence the assignment of tasks to workers. I consider the directed technical change problem in the first part of this section.

The counterpart to this problem is how workers can change their skills to better align themselves with the requirements of production. The second part of this section deals with the workers' skill accumulation problem. As before it is optimal to change the workers' skills (through training) so as to minimize the mismatch between tasks and workers.

1.3.1 Skill enhancing technology

Technical change can complement the current skills of workers. This is the case with the introduction of software that complements cognitive over manual skills in the completion of tasks, or heavy machinery, such as cranes, that complements dexterity over brawn. Unlike automation, this type of technical change affects the productivity of workers across tasks without displacing them. But, as with automation, technical change is followed by a re-assignment of tasks geared towards reducing the mismatch between workers and the tasks they perform. Workers who are more adept at the skills favored by the new technologies increase their productivity, while other workers lose their comparative edge. Changes in the boundaries of occupations are thus directed toward reducing the mismatch in the skills complemented by new technologies.

Just as with automation this type of technical change can also be directed. Which skills to favor depends on the joint distribution of workers and task, and the mismatch across different skills. In order to maximize production, it is optimal to weight more those skills for which mismatch is lowest, concentrating technology on enhancing the skills at which the workforce already excels. This contrasts with the way in which automation is directed. Instead of replacing workers at the tasks they are ill-suited for, technology enhances the worker's productivity by weighting the skills with a better match, while reducing the importance of the skills that the workforce lacks.

To make the discussion precise, I impose additional structure on how skill mismatch affects production. Consider two skills, cognitive and manual, and a production technology q as in (1.12). The relative importance of skills is then governed by matrix A, which I will assume to be diagonal taking the form:

$$A = \begin{bmatrix} \alpha & 0 \\ 0 & 1-\alpha \end{bmatrix}$$

where $\alpha \in [0,1]$. Higher α makes cognitive match more important for production, while simultaneously reducing the importance of manual skill match. The problem is to choose the value of α optimally to maximize output, taking into account the cost of changing technology and the changes in the assignment of tasks to workers:

$$\max_{\{T,\alpha\}} F(T,\alpha) - \Upsilon(\alpha) \qquad \text{s.t. } \forall_n\, D_n \leq p_n \tag{1.16}$$

The optimality condition for α can be obtained using the same techniques as before. The optimal α satisfies:

$$F(T,\alpha)(M_m - M_c) - \frac{\partial \Upsilon(\alpha)}{\partial \alpha} \geq 0 \tag{1.17}$$

Where M_s is total mismatch in skill s: $M_s = \sum_{n=1}^{N} \int_{\mathcal{Y}_n} (x_{n,s} - y_s)^2 \, dy$. The first term captures how much production would increase if α increases. The net gain in production is determined by the difference in total mismatch by skill, which depends on the assignment and the distribution of tasks and workers. If, for a given assignment, there is more mismatch in the manual dimension ($M_m > M_c$), the workforce is biased towards cognitive skills. It is then optimal to direct technical change towards cognitive skills by increasing α. In this way, technology reinforces the workforce's bias by giving more weight to skills for which there is a better match. The gain in output is balanced by the marginal cost of changing α. Absent that cost it is optimal to shift all the weight towards one of the skills. Specializing production to depend only on the skill with the lowest mismatch in the workforce.

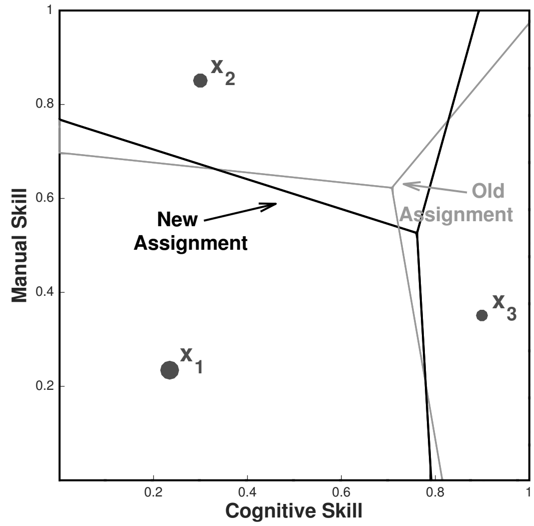


Figure 1.3: Example - Increase in the Weight of Cognitive Skills

Note: The figure shows the result of an increase in the weight of cognitive skills in a two-dimensional skill space (cognitive and manual skills). Three types of workers are considered $\{x_1, x_2, x_3\}$ with mass $P = \{0.5, 0.3, 0.2\}$. Tasks are uniformly distributed over the unit square, i.e $\mathcal{Y} = [0,1]^2$ and $g(y) = 1$ for all y. The production function q is given by (1.12) with $A = \text{diag}(\alpha, 1-\alpha)$, the value of a_x and a_y does not change the optimal assignment.

Figure 1.3 shows how the assignment of tasks to workers changes when the weight of cognitive skills increases. The boundaries of occupations shift and become less sensitive to differences in manual skills, discriminating across workers based on differences in their cognitive skills (as $\alpha \to 1$ the boundaries become vertical). As this happens workers' marginal product and substitutability change. Worker x_3 becomes less substitutable with others, as her cognitive skills differ from those of workers x_1 and x_2; recall from Proposition 2 that the elasticity of substitution decreases with the weighted distance between workers' skills. On the other hand, workers x_1 and x_2 become more substitutable, since they differ mostly in their manual skills, which are now less important in production.

These changes relate to observed patterns following the adoption of IT in production. There is a higher premium for workers with high cognitive skill (like college graduates), and a lower premium for manual intensive workers relative to low skill workers (Katz and Murphy, 1992). As shown in Section 1.2, the difference in marginal products (and compensation) across workers is a function of the differences in output at the boundaries. From equation 1.9:

$$\underbrace{\ln q(x_n, y) - \ln q(x_\ell, y)}_{\text{Diff. in Output}} = \underbrace{a_x'(x_n - x_\ell)}_{\text{Diff. in Skills}} - \underbrace{\left((x_n - y)' A (x_n - y) - (x_\ell - y)' A (x_\ell - y)\right)}_{\text{Diff. in Mismatch}}$$

When technology weights more cognitive skills, differences in cognitive skills are amplified through the mismatch term, while differences in manual skills are down-weighted. As a consequence, differences in marginal products and compensation become more influenced by differences in cognitive skills.

1.3.2 Worker training

The problem of worker training bears many similarities to the automation problem described in detail in Chapter 2, because of that I leave many of details to be developed later. Note first that the main question behind worker training, which skills should a worker have, is the same question behind the automation problem. The answer in both cases comes from the desire to reduce the mismatch between tasks and workers. The same way that the robot's skills are chosen to minimize the mismatch in the automated area, the worker's skills are chosen to minimize mismatch across the tasks in her occupation. Crucially, as the skills of the worker change the assignment will change, altering the tasks in the worker's occupation. Formally, the problem of optimal worker training is the same as that of choosing the robot's skills in (2.1), after appropriately modifying the cost function. Consider the problem of training worker n by choosing new skills $\tilde{x} \in \mathcal{S}$:

$$\max_{\{\tilde{x},T\}} \quad F\left(T,\tilde{x}\right)-\Gamma\left(\tilde{x}|x_n,p_n\right) \qquad \text{s.t. } \forall_{\ell}\, D_{\ell}\leq p_{\ell} \tag{1.18}$$

where the cost of changing skills (Γ) depends on the workers' current skills and mass. Following the same steps as in the automation problem, the first order condition of the problem is:

$$F\left(\lambda^{\star}\left(\tilde{x}\right),\tilde{x}\right)\int_{\mathcal{Y}_n}\frac{\partial\ln q\left(\tilde{x},y\right)}{\partial\tilde{x}}dG\left(y\right)-\frac{\partial\Gamma\left(\tilde{x}|x_n,p_n\right)}{\partial\tilde{x}}=0_{d\times1} \tag{1.19}$$

The first term captures the net gains in output from changing the workers' skills. The objective is to minimize skill mismatch across the tasks in the worker's occupation given the cost of changing the workers' skills. If q is given by (1.12) this is achieved by setting $\tilde{x}$ to the centroid of the occupation, and adjusting for the weight of skills in production (a_x) and the marginal cost of changing the worker's skills. Even if acquiring skills was costless, it is not always optimal to increase the worker's skills, doing so can generate its own costs as mismatch increases with respect to the boundary tasks of the worker's occupation. The problem is further complicated by the ambiguous effects on total output, since training one worker can induce higher mismatch for other workers, as the assignment changes. Because of this, condition (1.19) is only necessary, and not sufficient, for characterizing the optimal worker training.

The worker training problem is particularly useful when thinking about the introduction of new tasks. New tasks are likely to involve skills for which no worker is particularly well suited, inducing higher mismatch at early stages of adoption. It is then optimal to train workers to acquire skills that better match the changes in their occupations brought up by the new tasks.[20] The introduction of new technologies, like computers and IT, changes occupations by directly modifying the tasks carried out by workers. This is potentially a major disruption since the workforce is likely not to have the right combination of skills to perform the new tasks. This hurts the population groups who experience the highest mismatch while benefiting those whose skills align more with the new technology. In order to reduce the mismatch workers must train into new skills, more aligned with the new tasks. This training process will, in turn, modify occupations, changing the bundling of tasks and the roles of each type of worker in production.

1.4 Tasks and Unemployment

In Section 1.2 I assume that if a task is not assigned to any worker there is no output ($q(\emptyset, y) = 0$). This, together with the way output is aggregated into the final good (equation 1.3) implies that there is no production unless all tasks are assigned. An alternative approach is to consider the aggregation only across tasks which are performed, the ones assigned to a worker. In this way, it is possible to leave tasks unassigned without shutting down the production of the final good. A consequence of leaving tasks unassigned is that some workers are left unemployed. Which workers are unemployed, as well as the level of unemployment, depends on the assignment.

To formalize this idea consider an alternative to the final good technology described in equation 1.3:

$$F(T) = \exp\left(\int_{\mathcal{Y}\setminus\mathcal{Y}_{\emptyset}} \ln q(T(y), y)\, dG\right) - 1 \tag{1.20}$$

where the assignment T is extended so that tasks can be unassigned, i.e. $T : \mathcal{Y} \to \mathcal{X} \cup \{\emptyset\}$, and $\mathcal{Y}_{\emptyset}$ denotes the set of tasks left unassigned, i.e. $\mathcal{Y}_{\emptyset} = T^{-1}(\{\emptyset\})$. In this way, only the tasks that are assigned are considered in the aggregation. The level of the production needs to be adjusted since leaving tasks unassigned opens the possibility for a free lunch. If only

[20] This resembles the skill process of workers in Lise and Postel-Vinay (2015), where workers converge to the skill requirements of their occupations as they spend more time performing it.

a measure-zero set of tasks is assigned the integral in (1.20) is equal to zero, regardless of the assignment, and output is therefore 1. The subtraction takes care of this.

Its immediate that the result from the aggregation in (1.20) is equivalent to having $q(\emptyset, y) = 1$ in the original formula 1.3, extending T to take values over $\mathcal{X}$ and the unassigned option. That way tasks that are left unassigned (assigned to the empty set) don't add to the integral, obtaining the integral in (1.20) as a result. Adopting this convention turns out to be useful because it allows me to apply Proposition 1 in the same way as in Section 1.2. Leaving a task unassigned is equivalent to assigning it to a worker '$\emptyset$', which is in infinite supply, has an outside option of zero, and produces $q(\emptyset, y) = 1$ in all tasks.

The main difference with the results of Section 1.2 is that the level of the worker's outside option ($\underline{w}$) affects the assignment. To simplify calculations I will assume in this section that the outside option is given by a fraction of total output: $\underline{w}(T) = \underline{\lambda} F(T)$.[21] Under this assumption there exists a vector $\lambda^\star \in \mathbb{R}_+^N$ such that $\min \lambda_n^\star = 0$ and occupations are given by:

$$\mathcal{Y}_n = \{ y \in \mathcal{Y} \,|\, \forall_\ell \ \ln q(x_n, y) - \lambda_n^\star \geq \ln q(x_\ell, y) - \lambda_\ell^\star \quad \wedge \quad \ln q(x_n, y) - \lambda_n^\star \geq \underline{\lambda} \} \tag{1.21}$$

This is the equivalent to condition (1.6), it differs in the introduction of the second inequality, which compares the output of worker n in the task with the minimum payment the worker must receive. The second inequality comes from ensuring that it is profitable to assign the task; the outcome if the task is unassigned is $\ln q(\emptyset, y) = 0$, and the compensation of the worker depends on $\lambda_n^\star + \underline{\lambda}$. The unassigned tasks are:

$$\mathcal{Y}_\emptyset = \{ y \in \mathcal{Y} \,|\, \forall_n \ \ln q(x_n, y) - \lambda_n^\star < \underline{\lambda} \} \tag{1.22}$$

A necessary condition for a task to be assigned is that $q(x_n, y) \geq 1$, and the higher $\underline{\lambda}$ is, the fewer tasks are assigned for production.

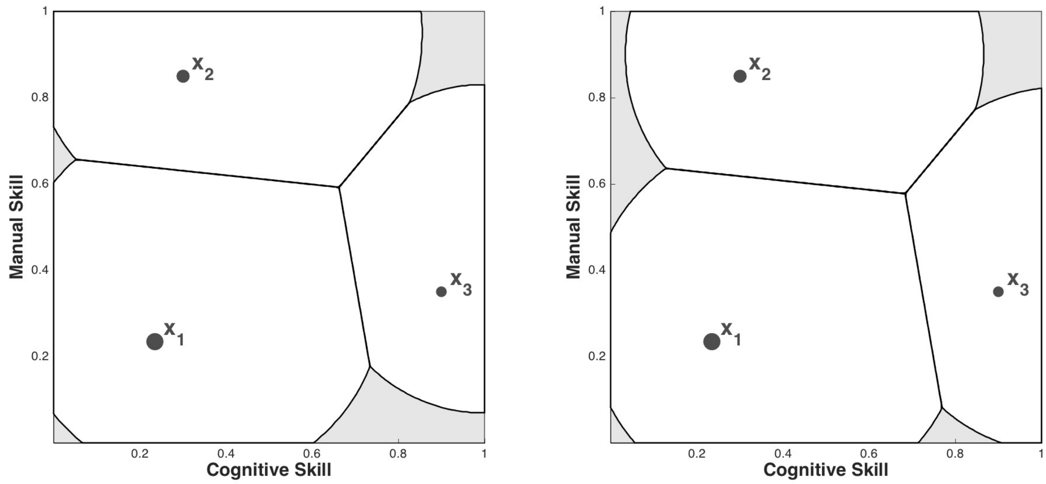
As in Section 1.2, the marginal product of a worker is given as in equation (1.8) and its compensation is given by $w_n = \lambda_n^\star F(T^\star) + \underline{w}$.

To fix ideas consider q as in (1.12), depending on the quadratic mismatch between worker and task's skills. Under that technology:

$$\ln q(x, y) = a_x^{'} x + a_y^{'} y - (x - y)^{'} A (x - y)$$

This provides a clear geometrical interpretation for which tasks are left unassigned. Workers

[21] Without this assumption it is not possible to determine the value of λ independently of the assignment T. The term $\underline{\lambda}$ in (1.21) has to be replaced by $\underline{w}/F(T)$.



(a) Symmetric Weights - $\alpha_y = 1/2$ (b) Asymmetric Weights - $\alpha_y = 3/4$

Figure 1.4: Assignment Example - Unemployment

Note: The figures show the assignment in a two-dimensional skill space (cognitive and manual skills). Three types of workers are considered $\{x_1, x_2, x_3\}$ with mass $P = \{0.5, 0.3, 0.2\}$. Tasks are uniformly distributed over the unit square, i.e $\mathcal{Y} = [0,1]^2$ and $g(y) = 1$ for all y. The production function q is given by (1.12) with $A = I_2$, $a_x = [0.2, 0.1]'$ and $a_y = \overline{a}_y [\alpha_y, 1-\alpha_y]'$, with $\overline{a}_y \in \mathbb{R}_+$ and $\alpha_y \in [0,1]$. The worker's outside option is 0.

will be assigned to a task only if it the mismatch is no greater than $a_x' x + a_y' y - \underline{\lambda}$.[22] This condition guarantees that enough output is produced by the worker for it to be profitable to perform the task and cover the worker's outside option. However, the condition does not imply that the task will be assigned to the worker, this depends on the comparison between workers' productivity as in Section 1.2 (see the first inequality in equation 1.21).

Which tasks to perform will depend critically on which tasks are more productive given current technology. This idea is captured by a_y, which determines which tasks generate more output, regardless of which worker performs them.[23] A higher cognitive weight in a_y makes cognitive intensive tasks more likely to be performed. For example, one of the effects of the increased use of information technology is to make cognitive intensive tasks more productive; as a consequence, it becomes optimal to perform more cognitive intensive tasks. Opposite changes can occur on the relevance of manual intensive tasks in production, shifting workers from manual to cognitive intensive tasks.

Figure 1.4 shows the optimal assignment in the model allowing for tasks to be unassigned,

[22] With $a_y = 0$, a task will be assigned only if it lies in a 'circle' of radius $\sqrt{a_x' x}$ around the skills of the worker. The shape of the 'circle' depends on the weights in matrix A.

[23] When all tasks must be performed the value of a_y does not affect the assignment. This is immediate from replacing (1.12) into 1.6.

and workers to be unemployed. The two panels differ on the weight of task skills in determining the output of a task, as measured by a_y. I assume that $a_y = \overline{a}_y \left[\alpha_y, 1-\alpha_y\right]'$ and I vary the relative importance of skills by choosing the weight $\alpha_y \in [0,1]$. A higher value of α_y makes cognitive intensive tasks more productive (tasks along the 45^o line do not change their productivity with α_y).

Panel 1.4a presents the assignment under equal skill weights in a_y. The grey areas represent unassigned tasks. It is optimal not to perform tasks for which agents have high mismatch, as in Section 2.2.1 these tasks are located along the boundaries of the task space, and the vertices of the assignment. Even though the wights on a_y are symmetric and the weights on a_x favor cognitive skills, most of the unassigned tasks involve high cognitive skills. This is because of the distributions of skills in the population. In the example, there are relatively few x_3 workers, and so, performing the high-cognitive tasks comes at the cost of a greater mismatch for workers x_1 and x_2, as the boundaries between them and x_3 would have to shift rightwards. Its worth noting that the assignment is such that only worker x_1 is unemployed. x_1 is the least productive worker type.

In Panel 1.4b the weights on skills change, making cognitive intensive tasks more productive, and manual intensive tasks less productive. As a response to this change workers x_1 and x_3 take over tasks in the bottom-right corner of the space, at the expense of tasks along the vertical axis. The higher productivity makes it worthwhile to reassign workers towards cognitive intensive tasks; doing so shifts the boundaries of $\mathcal{Y}_2$ towards x_1 and x_3, and away from the vertical axis. Unemployment is still concentrated in workers of type x_1.

The effect of the minimum wage $\underline{w}$ As in Section 1.2 the value of the workers' outside option $\underline{w}$ equals the minimum wage in the economy. But, unlike the problem in Section 1.2, the value of $\underline{w}$ affects the assignment. An increase in $\underline{w}$ increases wages, by increasing the minimum wage, and reduces employment, by limiting the set of tasks that are profitable to produce at the current wages.

The net effect on wages is nevertheless ambiguous. As the assignment of tasks changes so does the mismatch of workers at their boundary tasks. Mismatch necessarily goes down for the type of worker(s) that are not fully employed, but it might increase for other workers. Moreover, since wages reflect marginal productivities relative to the least productive workers (see equation 1.15), changes in the assignment can lead to a compression of the wage distribution. Much like with the introduction of automation (Section 2.2.1), the difference in productivity between the most and least productive workers can decrease. These effects must be weighted against the increase in the level of wages coming from $\underline{w}$ to determine the net effect on wages.

The effects on the wage distribution are similar to the ones documented for Brazil by Engbom and Moser (2018). An increase in the minimum wage will tend to compress the wage distribution by shifting up the level of wages from above, with small consequences for the workers with the highest productivity.[24] The total effect on employment depends on how productive workers are at their boundary tasks. An increase in the minimum wage will make some of these tasks unprofitable.

Automation and unassigned tasks

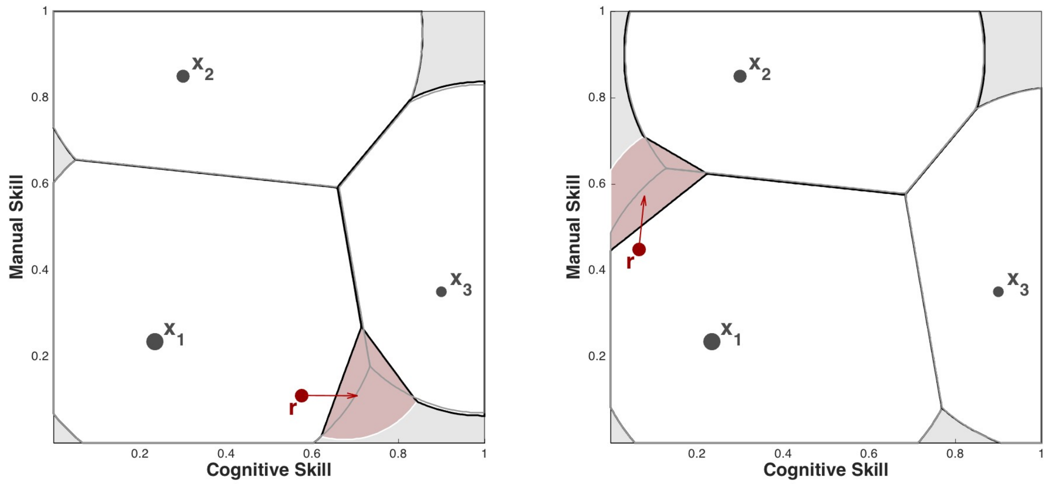
The characterization of automation as a worker replacing technology given in Section 2.2.1 changes once tasks can be left unassigned. It is now possible to direct automation towards the tasks which were previously unassigned, that is, tasks which are not worthwhile for workers to perform (because of low productivity), or tasks for which workers don't have the appropriate skills (high mismatch). If this happens, automation does not displace workers. Moreover, performing additional tasks necessarily increases output, potentially raising worker's marginal products and wages.

Whether or not it is optimal to automate unassigned tasks or to displace workers depends comparing the cost of automating a task with the mismatch in the worker-task assignment. Even though the mismatch is highest for unassigned tasks, it might be too costly to engineer the technology necessary to automate those tasks. In general, it will turn out to be optimal to automate tasks along the boundaries of occupations and unassigned tasks. As a result, automation ends up partially displacing workers. To illustrate this I expand the example in Figure 1.4 by solving the optimal automation problem. The results are presented in Figure 1.5. Production technology is the same for workers and the robot and is given by (1.12). The cost of automation is quadratic in skills as in the example in Figure 2.1.

Panels 1.5a and 1.5b present similar results, with the robot being placed so as to automate part of the cognitive/manual intensive tasks that were unassigned. The robot is only partially displacing workers since it is taking over unassigned tasks. Thus, the mass of unemployed workers increases, but less than the mass of tasks being automated (0.01 vs 0.03). Output increases due to the production of new tasks and the reduction in the mismatch in some of the old tasks.

The two panels in Figure 1.5 also show how the incentives for automation change as technology favoring the production of certain types of tasks change. If technological change favors cognitive intensive tasks over manual intensive tasks, workers are reassigned away from the

[24] The changes in mismatch are higher for the workers neighboring the lowest productivity worker.



(a) Symmetric Weights - $\alpha_y = 1/2$

(b) Asymmetric Weights - $\alpha_y = 3/4$

Figure 1.5: Assignment Example - Unemployment and Automation

Note: The figures show the assignment in a two-dimensional skill space (cognitive and manual skills). Three types of workers are considered $\{x_1, x_2, x_3\}$ with mass $P = \{0.5, 0.3, 0.2\}$. Tasks are uniformly distributed over the unit square, i.e $\mathcal{Y} = [0,1]^2$ and $g(y) = 1$ for all y. The production function q is given by (1.12) with $A = I_2$, $a_x = [0.2, 0.1]'$ and $a_y = \overline{a}_y [\alpha_y, 1-\alpha_y]'$, with $\overline{a}_y \in \mathbb{R}_+$ and $\alpha_y = 1/2$. The worker's outside option is 0. The automation cost function is: $\Omega(r) = r' A_R r$, with A_R diagonal. The mass of the robot is fixed at $p_r = 0.03$. The assignment without the robot is presented in grey.

latter and into the former (see Figure 1.4). Consequently, production can be increased by directing automation towards manual intensive tasks, in a way that disrupts the optimal assignment of tasks to workers the least as possible. In this scenario technological change makes new tasks available for workers while leaving other tasks unassigned, automation follows by taking over tasks that are no longer worthwhile for workers to perform. [25]

1.5 Concluding Remarks

I develop a framework to study occupations, where production takes place by assigning workers to tasks in a multidimensional setting. Occupations arise from the assignment process, instead of being taken as a preexisting feature of production. Because of this, the

[25] This idea is similar in spirit to Acemoglu and Restrepo (2018b)'s race between man and machine. As in their paper, changes in technology lead to a reassignment of workers towards more complex (and newer) tasks, while relatively simpler (and older) tasks are automated, displacing workers in the process. My model differs in a key aspect from theirs: the set of tasks to be performed is held fixed throughout. Yet, the environment I present can be reinterpreted, by considering the space of existing tasks to be larger than the set of tasks currently performed. Technological change, as well as changes in the skills of the workforce, continuously change the set of tasks that workers perform, moving towards more complex tasks (previously unassigned), and away from simpler tasks, that can then be automated.

framework incorporates endogenous changes in the boundaries of occupations in response to changes in the economic environment.

The model makes precise the role of tasks in defining the marginal product, compensation, and substitutability of workers. All these properties are shaped by how productive workers are at the tasks along the boundaries of their occupations. These are the tasks for which the workers are the least productive, and at which they are directly substitutable for other workers.

Chapter 2

Occupations and Automation in the U.S. Labor Market

2.1 Introduction

As an application of the framework developed in Chapter 1, I use the model to study the rise in automation observed in recent decades.[1] Automation technologies are directed towards replacing workers at specific tasks (e.g., industrial robots taking spots in the assembly line). Because of this, automation takes away some, but not all, of the tasks of an occupation. In a recent study, McKinsey Global Institute (2017) reports that while 50% of tasks are automatable using currently available technology, less than 5% of occupations are fully automatable. Consequently, automation is more likely to transform rather than to eliminate occupations. In the model, occupations are transformed directly by losing tasks to robots or software, and indirectly through the reassignment of tasks across workers.[2]

I model automation as a choice of the optimal size and location of a mass of identical robots in the task space. Robots replace workers at performing tasks. Automation can be directed through the location of the robots in the task space, which determines which tasks are automated. The optimal choice of location and mass weighs the cost of automation, which varies depending on the complexity of the tasks being automated, against the gains in output from replacing workers. Automation is thus directed towards regions that exhibit high skill-mismatch between workers and tasks. These regions are located around the boundaries of

[1] In manufacturing, Acemoglu and Restrepo (2017) estimate that industrial robots have displaced 756,000 workers between 1993 and 2007. Simultaneously, advances in software and AI have made it possible to automate tasks of clerical occupations and of more specialized workers like accountants.

[2] Other worker replacing technologies, like offshoring, operate through the same effects. See Blinder (2009) and Blinder and Krueger (2013) for offshorability measures based on occupational characteristics.

occupations (see Figure 1.1). Which boundaries are affected by automation depends on the gains in productivity relative to the cost of the robot.

As mentioned above, automation induces a reassignment across tasks. Because of this, the workers previously performing the automated tasks are not the only ones affected. It is optimal to reassign tasks so that only the workers with the lowest productivity are displaced by automation, preserving the employment of more productive workers. As a consequence of the reassignment, the mismatch between workers and tasks increases, potentially reducing workers' productivity and wages. As Acemoglu and Restrepo (2018a) point out, whether or not wages decrease depends on how productive robots are at the tasks they overtake. A major increase in productivity due to automation can increase workers' marginal product, increasing wages, while moderate increases in output in the automated tasks can be dominated by the higher mismatch experienced by workers, ultimately reducing their wages.

I estimate the model using occupational data for the U.S. labor market obtained from O*NET (the U.S. Department of Labor Occupational Characteristics Database) and the Bureau of Labor Statistics. Together, these data allow me to estimate the production technology and the distribution of skills across workers and tasks. I estimate the cost of automation using data on the automatability of occupations from Frey and Osborne (2017), and the cost of industrial robots from the International Federation of Robotics.

The model matches the wage structure across major occupational groups[3] and rationalizes observed trends in automation. I find that it is optimal to automate tasks with a high manual skill requirement, most of them related to manufacturing occupations such as metal workers and industrial machine mechanics. Yet, the displacing effects of automation fall mostly on workers who performed occupations requiring no education, such as food preparing and serving and building maintenance. In total, 4.1% of workers are displaced by automation in the model. I estimate the cost of automation to be $44,500 per replaced worker. This is equivalent to 2.6% higher than the the average wage in the sample. Automating tasks with a high manual skill requirement turns out to be optimal despite there being alternative tasks automatable at lower costs. The reason lies in the comparatively high mismatch between workers' skills and the skills demanded by the automated tasks.

The model also implies a decrease in wages following the automation of tasks. How much wages decrease depends on the reassignment of tasks across workers. How close the tasks of the worker's occupation are to the automated tasks determines how affected the worker's wages are. The increase in output from the automation of tasks is not enough to offset the

[3] I divide occupations into groups based on their 2-digit Standard Occupational Classification (SOC) and their skill requirements. See Section 2.3 and Appendix B.2 for details.

negative effects of the reassignment.

2.2 Theoretical Framework

I consider now technical change in the form of innovation in worker replacing technologies (such as robots, software, AI, offshoring). This form of technical change leads to the automation of tasks and the reassignment of (remaining) tasks to workers, and thus it is followed by changes in the role of workers in production, affecting their productivity and substitution patterns. The assignment also determines how substitutable workers are with alternative forms of production (e.g., robots, software).
As in the directed technical change studied in Chapter 1, automation can be directed towards specific tasks with the aim of increasing production. Production is increased the most by reducing the mismatch in between tasks and workers, by directing automation towards the tasks with the highest mismatch.

2.2.1 Directed automation

I introduce automation technology in the form of a robot that can replace workers in performing tasks.[4] The robot is modeled as a flexible technology that can be adapted to perform different types of tasks. This captures a key property of current technologies like industrial robots or advanced AI programs, which can be reprogrammed or adapted to carry out a variety of tasks (Acemoglu and Restrepo, 2017; Frey and Osborne, 2017). It also relates to other technologies, like offshoring, which, as automation, replace workers at the task they perform in their 'local' labor market (Blinder, 2009; Blinder and Krueger, 2013). The automation problem consists of designing a robot and optimally assigning tasks among the workers and the robot to maximize production. Tasks assigned to the robot are automated.
I treat the robot as a new type of worker. The key difference is that it is possible to choose the robot's skills and supply. I denote by $r \in \mathbb{R}^d$ the skills of the robot and by $p_r \geq 0$ its supply. The automation technology is embodied by a cost function $\Omega : \mathbb{R}^d_+ \times \mathbb{R}_+ \to \mathbb{R}$, so that the cost of producing a mass p_r of a robot with skills r is given by $\Omega\left(r, p_r\right)$. Many changes in the patterns of automation can be seen as changes in the cost of automation (Ω). For instance, recent advances in artificial intelligence are reducing the cost of automating

[4] I focus on the short-term effects of automation keeping the distribution of tasks fixed, abstracting from the potential gains from adding new tasks, or from performing more the existing tasks with the displaced workers. Acemoglu and Restrepo (2018b) study the effects of automation in an environment with changes in the set of tasks.

tasks intensive in cognitive skills (McKinsey Global Institute, 2017; Frey and Osborne, 2017), while previous innovations like the conveyer belt allowed for the automation of tasks involving manual skills.

Once the robot is designed the set of available workers is expanded to include it: $\mathcal{X}_R = \{x_1, \ldots, x_N, r\}$. Accordingly, the assignment is now described by a function $T_R : \mathcal{Y} \rightarrow \mathcal{X}_R$. The assignment of tasks to the robot will, of course, depend on how productive the robot is relative to the available workers. It is better to design robots so that they replace workers at tasks where skill mismatch is high, and worker productivity is low. These tasks are located along the boundaries of occupations. Automation is thus less likely to occur at 'core' tasks of an occupation, for which the worker is best suited. I denote by $q_R : \mathbb{R}^d \times \mathcal{Y} \rightarrow \mathbb{R}$ the production technology of the robot so that a robot r performing task y produce $q_R(r, y)$.

When tasks are automated the total demand for labor decreases,[5] inducing unemployment among workers. Which workers become unemployed depends on the way in which the assignment reacts to the introduction of the robot. As tasks are assigned to the robot, the workers who would have performed those tasks are directly displaced. Yet, these workers do not necessarily become unemployed since they can take over the tasks of other workers. The end result of this process depends on the substitutability and relative productivities of the workers in the economy. It is the workers with the lowest marginal product who will become displaced (unemployed) as a response to the introduction of the robot, even if the tasks in their occupation are not directly affected by automation. This follows from the order in which workers are substituted from one another described in the previous section when discussing the marginal product of workers.

The automation problem is to choose jointly the skills and mass of the robot (r, p_r), and the new assignment (T_R) to maximize output, net of the automation cost (Ω):

$$\max_{\{r, p_r, T_R\}} \quad F_R(T_R, r) - \Omega(r, p_r) \qquad \text{s.t. } \forall_n\, D_n \leq p_n \qquad D_R \leq p_r \tag{2.1}$$

where:

$$F_R(T_R, r) = \exp\left(\int_{\mathcal{Y} \setminus \mathcal{Y}_R} \ln q\left(T_R(y), y\right)\, dG + \int_{\mathcal{Y}_R} \ln q_R(r, y)\, dG\right) \tag{2.2}$$

and

$$\mathcal{Y}_R = T_R^{-1}(r) \qquad D_R = \int_{\mathcal{Y}_R} dG$$

It is convenient to think of the problem in two steps, first solving for an optimal assignment

[5] This is a consequence of the assumption that the set of tasks to be performed $(\mathcal{Y})$ is fixed, as is the distribution of tasks (G).

given a set of workers and a robot, and then choosing the optimal skills and mass of the robot taking into account the effect on the optimal assignment. In this way, the problem of finding an optimal assignment can be simplified making use of the results in Proposition 1. Taking as given the robot skills and mass (r, p_r), the optimal assignment is necessarily characterized by a vector $\mu^{\star} \in \mathbb{R}^{N+1}$:[6]

$$T_R(y) = x_n \longleftrightarrow \forall_{\ell} \ \ln q(x_n, y) - \mu_n^{\star} \geq \ln q(x_\ell, y) - \mu_\ell^{\star} \qquad (2.3)$$
$$\wedge \quad \ln q(x_n, y) - \mu_n^{\star} \geq \ln q_R(r, y) - \mu_R^{\star}$$

This assignment satisfies the capacity constraints of all robots and the workers.
Abusing notation the problem is then:

$$\max_{\{r, p_r\}} \quad F_R(\mu^{\star}(r, p_r), r) - \Omega(r, p_r) \qquad (2.4)$$

where $\mu^{\star}$ depends on the value of r and p_r, and takes into account how the optimal assignment reacts to changes in the robot skills and mass. The first order conditions of the problem are now derived using the envelope theorem of Milgrom and Segal (2002) and Reynolds' transport theorem (Theorem 2 in the Appendix):[7]

$$\nabla\left(F_R(\mu^{\star}(r, p_r), r) - \Omega(r, p_r)\right) = \nabla F_R(\mu^{\star}, r) - \nabla\Omega(r, p_r) = 0_{d+1\times 1}$$

I first focus on the the derivative of output with respect to the robot's skills:

$$\frac{\partial\left(F_R(\mu^{\star}(r, p_r), r) - \Omega(r, p_r)\right)}{\partial r} = F_R(\mu^{\star}(r, p_r), r) \int_{\mathcal{Y}_R} \frac{\partial \ln q_R(r, y)}{\partial r} dG - \frac{\partial \Omega(r, p_r)}{\partial r} = 0_{d\times 1} \qquad (2.5)$$

The marginal cost of changing the robot's skills is balanced with the gain in output the change in skills induces.
The first term in (2.5) accounts for the change in output across all tasks assigned to the robot. Changing the robot's skills changes the productivity of the robot across tasks, in general increasing it for some tasks and decreasing it for others, as changing r can increase

[6]This strategy has been exploited extensively by the optimal sensor placement literature under quadratic loss functions. Under that loss function the optimal assignment is necessarily a power diagram, see Aurenhammer et al. (1998, Thm . 1) and Xin et al. (2016, Thm. 1).

[7]See Xin et al. (2016) for further applications in the theory of optimal power diagrams with capacity constraints. Proposition 3 in Appendix A.2 provides an alternative derivation for the result based on Goes et al. (2012). The alternative proof is more tedious, but being more explicit it makes it clear how changing the robot's skills affects output.

mismatch for some of the tasks in $\mathcal{Y}_R$. Thus, the first term gives the net gain in output from a change in the robot's skills. Unlike previous results, all of the tasks assigned to the robot matter, and not only those in the boundary of the automated region.

It is convenient to use an explicit functional form for q_R to fix ideas. Take for instance the production functioned presented in (1.12). Assuming that $q_R(r,y) = q(r,y)$, the first order condition can be expressed as:

$$\frac{\partial\left(F_R\left(\mu^\star\left(r,p_r\right),r\right)-\Omega\left(r,p_r\right)\right)}{\partial r} = 2F_R D_R\left(\frac{a_x}{2} - A\left(r - b_R\right)\right) - \frac{\partial\Omega\left(r,p_r\right)}{\partial r} = 0_{d\times 1}$$

where $b_R = \frac{\int_{\mathcal{Y}_R} y dG}{D_R}$ is the centroid (or barycenter) of the automated area. Absent other considerations it is optimal to set the robot's skills to the centroid of the automated region, this minimizes the (quadratic) loss from skill mismatch, thus maximizing the robot's output.[8] The robot's skills deviate from the centroid to account for gains from having higher skills (a_x), and for the cost of automation ($\partial\Omega(r,p_r)/\partial r$).

The first order condition with respect to p_r takes the usual form of equating marginal product to marginal cost. As in (1.8), the marginal product is $\text{MP}_R = F_R\mu_R^\star$:

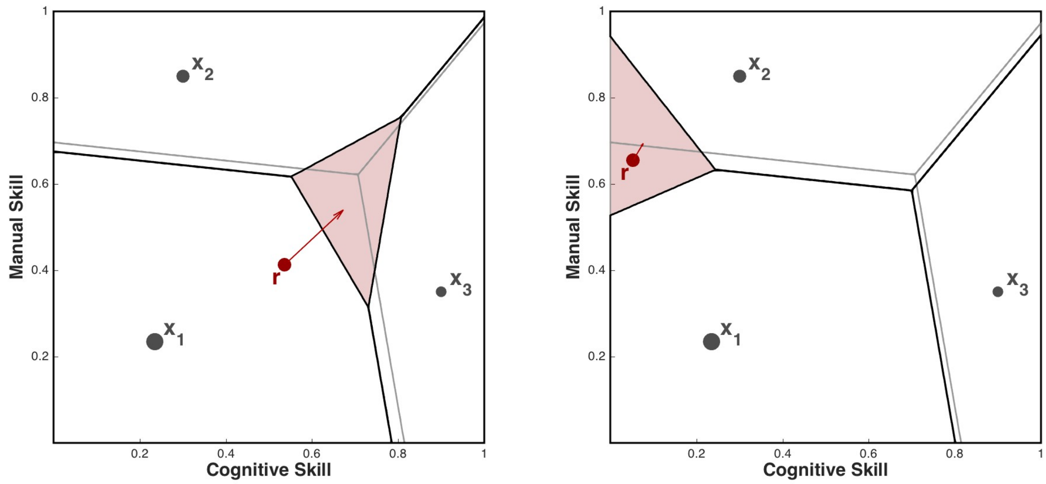
$$\frac{\partial F}{\partial p_r} = F_R\left(\mu^\star\left(r,p_r\right),r\right)\mu_R^\star - \frac{\partial\Omega\left(r,p_r\right)}{\partial p_r} = 0 \tag{2.6}$$

Note, however, that the automation problem in (2.1) is not concave in r and thus condition (2.5) is only necessary and not sufficient (Urschel, 2017). The first order condition is descriptive of the properties that the robot's skills must satisfy relative to the automated region, but it does not pin down the set of tasks to be automated. Regardless, the problem can be solved numerically using a version of Lloyd's algorithm (Lloyd, 1982). The algorithm consists of finding the optimal assignment for a given value of r and p_r, then adjusting r and p_r to satisfy their respective first order conditions. The process is repeated until convergence. This algorithm has been proven to converge monotonically to a local minimum of the objective function (see Du et al. (2010) and references therein). Urschel (2017) gives sufficient conditions that can be checked for convergence to a global minimum.[9]

Figure 2.1 presents the solution to the automation problem assuming that q and q_R are given by (1.12), and that the automation cost is quadratic in the robot's skills: $\Omega(r) = r' A_R r$.

[8] This result is shared by the literature on the optimality of centroidal Voronoi diagrams and is exploited extensively in optimal sensor placement problems. It is also linked to K-means and other vector quantization methods.

[9] In practice there are only finitely many candidates for a global minimum, making the selection of the solution simple. It is optimal to automate tasks around one of the vertices of the partition induced by the initial assignment (without automation). Aurenhammer (1987) shows there at most 2n-5 of these vertices in a diagram when the production function is quadratic in x and y, $d = 2$ and $n \geq 3$.



(a) Symmetric weights

(b) Asymmetric weights

Figure 2.1: Directed Automation Example - Quadratic Automation Cost

Note: The figures show the result of the automation problem taking the robot's mass as given in a two-dimensional skill space (cognitive and manual skills). Three types of workers are considered $\{x_1, x_2, x_3\}$ with mass $P = \{0.5, 0.3, 0.2\}$. Tasks are uniformly distributed over the unit square, i.e $\mathcal{Y} = [0,1]^2$ and $g(y) = 1$ for all y. The production function q is given by (1.12) with $A = I_2$, $a_x = [0.2, 0.1]^{'}$ and $a_y = [0, 0]^{'}$. The automation cost function is: $\Omega(r) = r^{'} A_R r$, with A_R diagonal. The mass of the robot is fixed at $p_r = 0.05$. The assignment without the robot is presented in grey.

The two panels differ only on the weights of cognitive and manual skills in the automation cost function. To keep the example simple I fix the mass of the robot exogenously.

Panel 2.1a assumes symmetric weights. It is then optimal to automate the tasks around the center vertex of the original assignment (without the robot). These are the tasks with the highest mismatch. Yet, because of the cost of endowing the robot with high cognitive and manual skills, it is not optimal to have placed the robot's skills in the automated area. The introduction of the robot displaces all three workers from the tasks being automated, but not all types of workers become unemployed. The assignment reacts endogenously to the automation, favoring the more productive workers (x_2 and x_3) over the least productive worker (x_1). The boundaries of the occupations adjust, re-assigning tasks along the boundaries of $\mathcal{Y}_1$ to workers of type x_2 and x_3. Only x_1 is displaced after tasks are reassigned.

Panel 2.1b assumes asymmetric weights, with a higher weight on automating cognitive tasks. It is no longer optimal to automate the tasks in the center vertex due to the high cost of automating cognitive skills. Nevertheless, the automated tasks are still located along the boundary of occupations. In this case around the vertex formed by $\mathcal{Y}_1$, $\mathcal{Y}_2$ and the boundary of the task space. Since these tasks involve less cognitive skills it is possible to locate the robot's skills closer to the centroid of the automated region. As in panel 2.1a automation takes away tasks from workers, in this case only from x_1 and x_2. The assignment reacts to automation by reassigning tasks along the boundary of $\mathcal{Y}_1$ towards more productive workers. Again, x_1 is the only type of worker displaced by automation.

The two examples in Figure 2.1 capture a general feature of the automation problem: it is optimal to automate tasks around the vertices of the original assignment (without the robot) since those are the tasks with the highest mismatch. Which tasks are optimally automated is jointly determined by the original assignment and the properties of the cost function.

Wages and the labor share The effect of automation on wages is ambiguous. First, automation induces a reassignment across tasks. Because of this, the workers previously performing the automated tasks are not the only ones affected. The reassignment weakly increases the mismatch between workers and tasks. It is easy to show that introducing the robot relaxes the assignment problem, and weakly decreases the value of the multipliers $\lambda^\star$ associated with each worker. The increase in the mismatch (reduction of $\lambda^\star$) reduces wages. Second, automation reduces the skill mismatch for the tasks being automated, increasing overall output. This increases the marginal product of all workers, and thus their wages.

As Acemoglu and Restrepo (2018a) point out, whether or not wages decrease depends on how productive robots are at the tasks they overtake. A major increase in productivity due to automation can increase workers' marginal product, increasing wages, while moderate increases in output in the automated tasks can be dominated by the higher mismatch experienced by workers, ultimately reducing their wages.

Regardless of the change in wages, the labor share decreases because of the decrease in $\lambda^\star$. The labor share is given by:

$$\text{LS} = \frac{\sum_{n=1}^{N} w_n D_n}{F\left(T\right)} = \sum_{n=1}^{N} \lambda_n^\star D_n + \frac{w}{F\left(T\right)} G\left(\mathcal{Y}\right) \tag{2.7}$$

Both terms decrease with automation. Higher mismatch for workers reduces $\lambda^\star$, lower demand for workers reduces D, and the second term decreases as output increases.

2.3 Empirical Application

I now use the model developed in Section 1.2 to examine U.S. occupational data. First I estimate the model using data on occupation characteristics and wages. I then use data on the automatability of occupations to infer the cost of automation. Finally, I make use of the results in Section 2.2.1 to solve for the optimal direction of automation.

Data sources The main source of data for the estimation of the model is the 2010 version of O*NET.[10] The O*NET is the U.S. Department of Labor Occupational Characteristics Database, it contains information on attributes of 974 occupations. Attributes characterize the knowledge, skills, and abilities that are used to perform the tasks that make up an occupation. The data reports the importance of 277 such attributes, as rated by analysts with expertise in each occupation.[11]

I complement the O*NET data with tabulations from the 2010 Occupational Employment Statistics (OES), provided by the Bureau of Labor Statistics.[12] The OES include data on employment and average annual wages by occupation for non-military occupations. It covers a total of 796 occupations with total employment of 127.1 million workers.

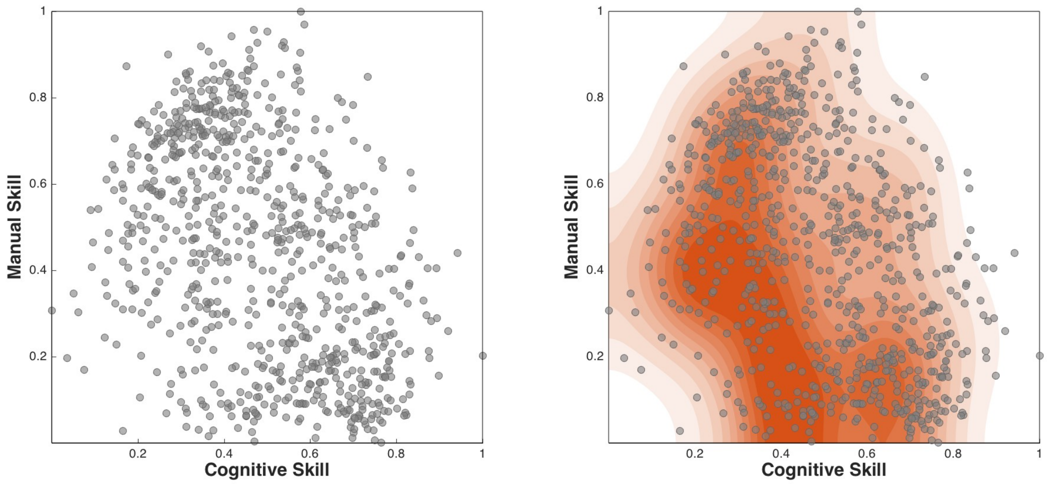
I merge the two datasets matching the SOC and title of each occupation. The resulting data contains 800 occupations with total employment of 119 million workers. The discrepancy between the number of occupations in the original OES data and the final sample I use is explained by the higher detail of occupations in the O*NET data. In particular, the OES data lumps smaller or specialized occupations into an 'all other' category. I am able to match these categories to individual occupations contained in the O*NET sample. The loss of employment is also explained by the 'all other' category. Not all of these occupations have a counterpart in the O*NET sample, unmatched occupations are dropped from my sample since I don't observe any of their attributes.

Occupations' skill requirements To obtain a measure of the skill requirements of each occupation I proceed in two steps, similar to the ones used by Guvenen et al. (2015b) and Lindenlaub (2017). First, I categorize attributes into skill groups. Second, I reduce the dimension of each group by taking the first principal component of the group of attributes

[10] Data is available at: https://www.onetcenter.org/db˙releases.html

[11] My data does not allow me to address the variation in task composition and skill requirements within occupations documented in Autor and Handel (2013) and Stinebrickner et al. (2019). Yet, since my analysis is aggregate as shown in the rest of this section, this should not impede obtaining valuable conclusions out of the exercise.

[12] Data is available at: https://www.bls.gov/oes/tables.htm



(a) Occupation Skill Measure

(b) Tasks' Skill Requirement Distribution

Figure 2.2: Occupation Skill Measures and Task Distribution

Note: The left panel shows the cognitive and manual skill components of the 800 occupations in my sample. Each point in the graphs corresponds to one occupation. Occupational skill requirements are computed respectively as the first principal component of cognitive and manual occupational attributes.
The right panel shows the level curves of the distribution of tasks' skill requirements, inferred from the distribution of skill components of occupations, weighted by employment.

as my measure of skill.[13] For the exercise below I will only consider two skill groups, namely cognitive and manual skills. In total, I use 69 cognitive attributes, and 47 manual attributes, the complete list of attributes is reported in Appendix B.1. Figure 2.2a shows the skill measure of the 800 occupations in the sample in the cognitive-manual skill space.[14]

Distribution of tasks I take the distribution of skill requirements across occupations as informative of the underlying (continuous) distribution of skill requirements across tasks. I construct a non-parametric estimate of the distribution of skill requirements (g) by smoothing the (weighted) distribution of skill requirements of occupations with a Gaussian kernel. The level curves of the resulting function are presented in Figure 2.2b. Darker regions have a higher density.

Distribution of workers I use occupational data to infer the distribution of workers in the economy. To do so I group the occupations in my sample into five categories based on

[13] I weight occupations by employment when performing principal component analysis on them. The results are not noticeably affected if I do not weight occupations. I have also repeated this exercise by focusing on 6 hand-picked attributes for each skill group and using their average value as the measure of skill. There is no change in the general distribution of skills.

[14] The correlation between the constructed measure of skills is -0.23.

the Standard Occupational Classification (SOC) 2-digit code and the skills requirements of the occupations. The list of occupations in each category is presented in Appendix B.2. Figure 2.3a presents the occupations in the sample grouped by educational requirement. The first group, composed by occupations in food serving and preparation, personal care and building maintenance is characterized by relatively low cognitive requirements and an intermediate level of manual requirements. These occupations require (in general) no education.[15] As a group these occupations account for 19% of employment in the sample. The second group is composed by sales personnel, office workers and clerks. These occupations are characterized by relatively low cognitive and manual requirements and is accounts for 28% of the employment. The third group is mostly composed by manufacturing occupations and is characterized b ya high manual requirement. It also includes transportation and repair occupations. It accounts for 22% of employment. The last two groups are composed by professional occupations, such as doctors, lawyers, managers, educators, scientists and engineers. The groups are divided based on the manual requirements of the occupations, respecting the 2-digit SOC classification. They account for 14% and 17% of employment respectively. Figure 2.3b highlights occupations sample occupations of each group.

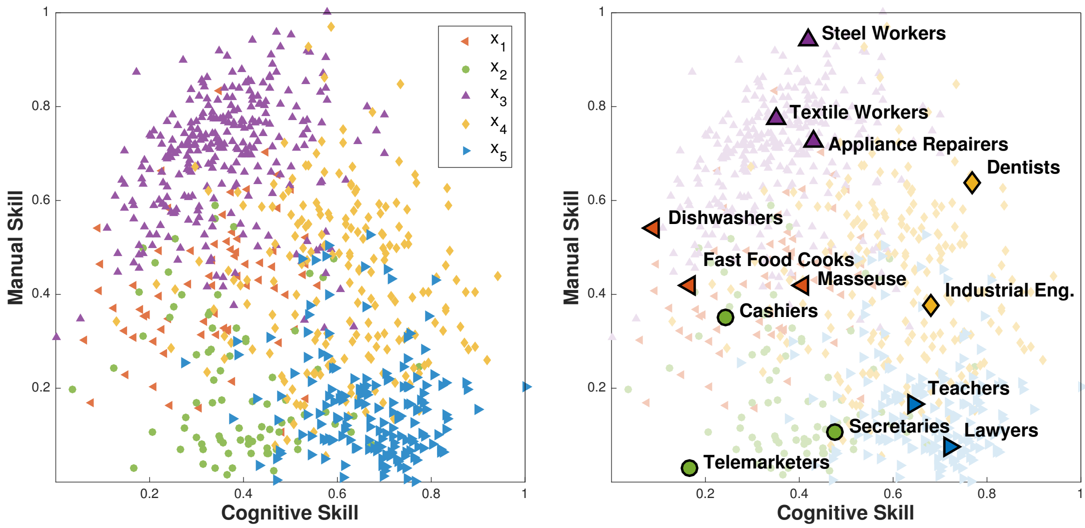
I interpret each category of O*NET occupations as representing the tasks performed by a type of worker. I infer the mass (p_n) and skills (x_n) of each type of worker from the occupations in its category. Thus, the mass of each type of worker is the share of employment of the occupations in the category; the skills of each type of worker are obtained as the employment-weighted average of the skill requirements of the occupations in the category.[16] Figure 2.4 presents the resulting distribution of workers in the skill space. Each worker should be interpreted as the average worker of the occupations in each category.

Production technology and assignment Task output is parametrized as in (1.12) with $\alpha_y = [0\,,\,0]^{'}$ and A a diagonal matrix:

$$q\,(x, y) = \exp\left(a_x^{'} x - (x-y)^{'}\,A\,(x-y)\right) \qquad \text{where: } A = \begin{bmatrix} A_{cc} & 0 \\ 0 & A_{mm} \end{bmatrix} \tag{2.8}$$

[15] The O*NET includes information on the education requirement of occupations, classifying each in one of five categories: less than high-school, high-school, vocational training (trade-school), college and post-graduate education. The code for the educational requirement of the occupation is 2D1. For occupations with missing educational requirement I use O*NET's job zone classification to impute the value. Job zones are a mixture of educational and experience requirements for an occupation. The five job zone categories overlap to a great extent with the educational requirement variable.

[16] Other studies, like Lindenlaub (2017), also use O*NET's occupational skill requirements as a measure of workers' skills. In contrast, studies like Lise and Postel-Vinay (2015) and Guvenen et al. (2015b) use worker-side data from the NLSY to compute workers' skills.



(a) Educational Requirement Categories (b) Sample Occupations by Category

Figure 2.3: Occupations by Educational Requirement Categories

Note: The left panel shows the cognitive and manual skill components of the 800 occupations in my sample by category. Each point in the graphs corresponds to one occupation. Occupational skill requirements are computed respectively as the first principal component of cognitive and manual occupational attributes. The right panel highlights sample occupations in each group.

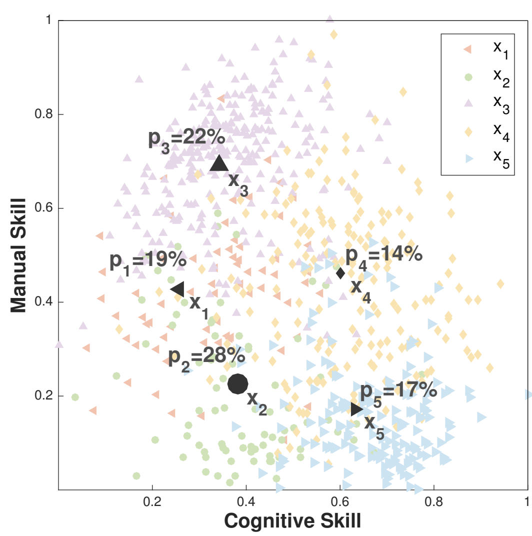


Figure 2.4: Distribution of Workers by Educational Requirement Categories

Note: The figure shows the estimate for worker's skill (x_n) and mass (p_n) and the underlying occupations in the five categories. Worker's skill (x_n) is computed as the employment-weighted average of the cognitive and manual skill requirements of the occupations in the worker's category. Worker's mass (p_n) is computed as the employment share of the occupations in the worker's category.

There are then four parameters to estimate characterizing the role of mismatch in production (A_{cc} and A_{mm}) and the effect of skills in wages (a_x). I estimate the parameters in a two-step procedure. First choosing the ratio A_{cc}/A_{mm} to minimize the classification error between the model's assignment and the data over the category of occupations. Then, the scale of the mismatch (A_{mm}) and the value of a_x are estimated to match wages by occupational category.

Recall from equation (1.13) that under this technology the boundaries of occupations are given by hyperplanes, whose normal vectors are defined by A. Thus the ratio A_{cc}/A_{mm} fully determines the assignment given the estimated distribution of tasks' skill requirements (g), and the distribution of workers (x_m, p_n). For a given value of A_{cc}/A_{mm} it is possible to classify each occupation in the sample according to the type of worker it is assigned to. I choose the value of A_{cc}/A_{mm} to minimize the classification error between the model's assignment and the observed educational requirement of the occupation.

Having estimated A_{cc}/A_{mm} it is possible to use wage data to estimate the remaining parameters. To do this, we first relate the value of A_{mm} and a_x to the multipliers λ associated with the optimal assignment. From (1.13) it is possible to obtain an equation for λ_n as a function of skills and mismatch with respect to the lowest paid worker:

$$\lambda_n = a_{x'} \left(x_n - \underline{x}\right) - \underbrace{\left(x_n - y_n\right)' A \left(x_n - y_n\right)}_{x_n \text{ mismatch}} + \underbrace{\left(\underline{x} - \underline{y}\right)' A \left(\underline{x} - \underline{y}\right)}_{\underline{x} \text{ mismatch}} \tag{2.9}$$

where $\underline{x}$ are the skills of the lowest paid worker, and y_n and $\underline{y}$ are boundary tasks of workers x_n and $\underline{x}$ respectively.

The second and third terms in (2.9) give the mismatch of workers x_n and $\underline{x}$ at the boundaries of their occupations. Although mismatch is not directly observable, it can be backed out using only the estimate of A_{cc}/A_{mm} since it determines the boundaries of the assignment. Its easy to show from (1.13) that the multipliers of the assignment with $a_x = \vec{0}$ and $A^0 = \operatorname{diag}\left(A_{cc}/A_{mm}, 1\right)$ provide an exact measure of the mismatch terms in (2.9):

$$\begin{aligned} A_{mm}\left(\lambda_n^0 - \underline{\lambda}^0\right) &= A_{mm}\left(\left(\underline{x} - \underline{y}\right)' A^0 \left(\underline{x} - \underline{y}\right) - \left(x_n - y_n\right)' A^0 \left(x_n - y_n\right)\right) \\ &= \left(\underline{x} - \underline{y}\right)' A \left(\underline{x} - \underline{y}\right) - \left(x_n - y_n\right)' A \left(x_n - y_n\right) \end{aligned}$$

where λ^0 is the multiplier vector of the auxiliary assignment and $\underline{\lambda}^0$ is the multiplier of the lowest paid worker.[17]

[17] No that even though $\underline{\lambda} = 0$ it is not necessarily the case that $\underline{\lambda}^0 = 0$. λ^0 measures only the mismatch along the boundaries relative to the worker with the highest mismatch. $\lambda^0 = 0$ only for the worker(s) with

Table 2.1: Estimates of Production Technology

A_{cc}	A_{mm}	a_x^c	a_x^m
1.7	2.8	1.7	0.15

Note: Estimated values for the parameters of the production technology in (2.8). A_{cc} and A_{mm} give the weight of cognitive and manual mismatch respectively. a_x^c and a_x^m give the weight of worker's cognitive and manual skills on marginal products and wages.

Finally, it is possible to construct an empirical measure of the multipliers λ from (1.10):

$$\hat{\lambda}_n = \frac{w_n - \min_\ell (w_\ell)}{F(T)}$$

where the value of κ is pinned down by the lowest observed wage since $\min_\ell (\lambda) = 0$, and $F(T)$ is a measure of total output.[18] Wages are taken as the employment-weighted average of the average annual wage across occupations in each educational requirement category. The estimates of A_{mm} and a_x are then obtained from the fitting the following linear relation:

$$\hat{\lambda}_n = a_{x'} (x_n - \underline{x}) + A_{mm} \left(\lambda_n^0 - \underline{\lambda}^0\right)$$

Table 2.1 shows the estimates for the parameters of the production technology for task output, and Figure 2.5 shows the assignment given those estimates. The value weight on cognitive mismatch (A_{cc}) is 0.59 times the weight on manual mismatch (A_{mm}). The value of a_x reflects the pattern of higher wages for more cognitive demanding occupational categories, thus having $a_x^c > a_x^m$. An increase in cognitive skill of 0.01 implies an increase in wages of 1.7% more of total output, compared to a 0.15% increase for an equal increase in manual skills.

Table 2.2 presents the wage of each category in the data and the ones implied by the assignment under the estimated task output technology. The wage of the first category is matched by construction by setting κ so as to match it. The other wages are obtained as in (1.10). In general, wages in the model overestimate the level of observed wages, but differences are small.

the highest mismatch. In contrast, λ also includes the importance of worker's skills in production, measured by a_x. The worker with the highest mismatch is not necessarily the lowest paid worker if her skills are high enough.

[18]Following Karabarbounis and Neiman (2014) I take output as corporate sector value added, measured as 60% of GDP for 2010. This implies a labor share of 62% for the full BLS sample, and 57% for my matched O*NET sample. The difference is explained by the loss of employment and the lower average wage of my sample, \$42000 compared with \$44000 2010 dollars.

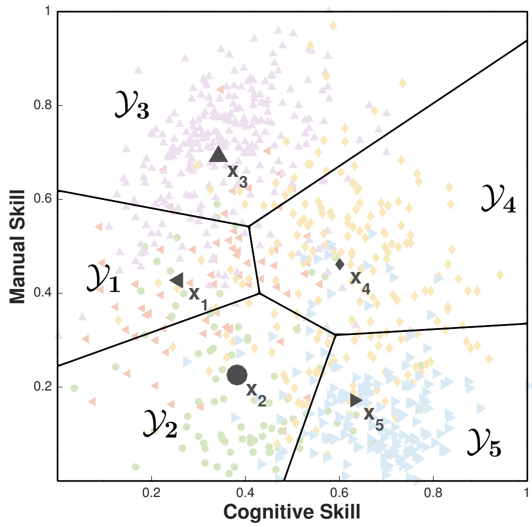


Figure 2.5: Assignment

Note: The figure shows the assignment of tasks to workers in the cognitive-manual skill space given the output task technology in (2.8) and the parameter estimates in Table 2.1.

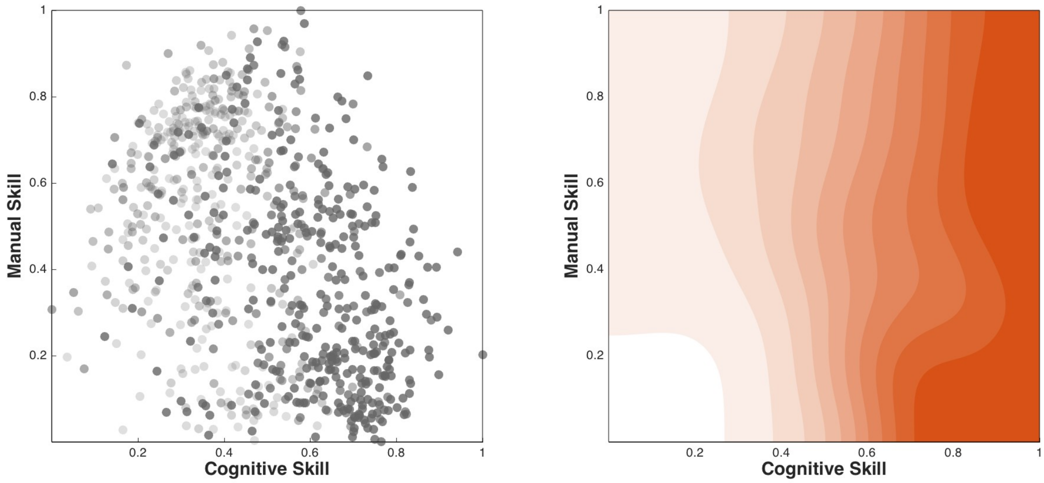
Table 2.2: Wages in the estimated model

Wages	x_1	x_2	x_3	x_4	x_5
Data	23.4k	34.1k	36.9k	68.3k	68.3k
Model	23.4k	34.3k	37.6k	67.6k	68.7k

Note: The table presents the average annual wages in 2010 dollars across occupations in each educational requirement category, and the implied wages for each worker type under the assignment shown in Figure 2.5.

Figure 2.6: Cost of Automation

(a) Automatability Index by Occupation (b) Cost of Automation - Level Curves



Note: The left panel shows the occupations in Frey and Osborne (2017)'s sample discriminated by their automatability index. The index measures the likelihood that an occupation is automatable given current technology. Darker occupations correspond to lower indices of automatability, or occupations which are less likely to be automatable.
The right panel shows the level curves of the cost function inferred from the automatability index of occupations. Darker regions correspond to higher automation costs.

Cost of automation Occupations vary in how automatable they are depending on the combination of skills that are required to perform the tasks that compose them. I use estimates of the automatability of occupations provided by Frey and Osborne (2017) to estimate the cost of automation.[19] The main assumption is that the cost of automation is inversely related to the degree of automatability of an occupation. Figure 2.6a shows the occupations in my sample with darker points reflecting lower indices of automatability. I infer the shape of the cost function non-parametrically by fitting a Gaussian kernel to Frey and Osborne (2017)'s automatability index on the cognitive-manual skill space. Figure 2.6b shows the level curves of the estimated cost function.

I scale the automation cost function to match the average cost of an industrial robot per replaced worker to the cost of automation in manufacturing occupations (SOC code 51). I obtain the average cost of an industrial robot from the International Federation of Robotics (IFR) annual report.[20] The cost is \$147,883 for 2010. I take the worker replacement ratio from Acemoglu and Restrepo (2017) who estimate that an industrial robot replaces between

[19]Frey and Osborne (2017) provide an index of automatability for the occupations in the 2010 O*NET based attributes related to 'computerization bottlenecks'. The index goes from 0 to 1 and gives the likelihood that an occupation is fully automatable given its attributes.

[20]See the executive summary in https://ifr.org/free-downloads/

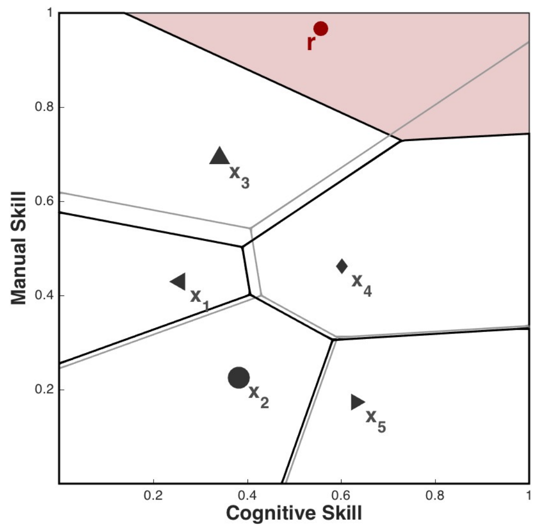


Figure 2.7: Assignment with Optimal Robot Placement

Note: The figure shows the assignment of tasks to workers and the robot in the cognitive-manual skill space given the estimated automation cost. The shaded region corresponds to automated tasks, it accounts for 4.1% of tasks. x_1 workers are displaced by automation.

4 and 6.2 workers. I take their preferred estimate of 5.1. Finally, I assume that the cost function is linear in the mass of the robot.

Automation problem Figure 2.7 shows the assignment under the optimal robot placement. It is optimal to automate manual intensive task along the upper edge of the skill space, placing the robot at $r = [0.56\,,\,0.97]$. The automated region accounts for 4.1% of the tasks ($p_r = 0.041$), displacing the same share of workers.[21] The cost of the robot is \$44,500 per unit of replaced-workers. Output increases 2.3% as a result of the decrease in the mismatch.

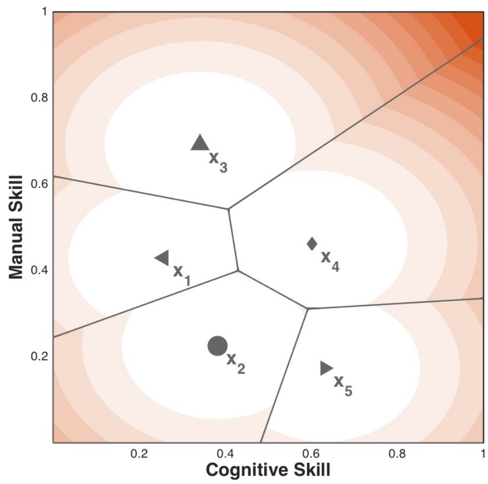
Even though the automated tasks are those with high manual requirements, mostly in the region of manufacturing, repair and installing occupations (under the third category), it is the workers of type x_1 who are displaced under the new assignment. In order to keep workers x_3 employed the boundaries of occupations change, and as a consequence wages decrease, reflecting the drop in marginal productivity of workers relative to x_1 workers. The change in wages is presented in Table 2.3. Recall from (1.9) that marginal products are given by differences in productivity across workers. As the assignment changes and workers with lower productivity are displaced, two effects come into play. First, the mismatch at the boundaries decreases for the displaced workers. Second, the new assignment implies higher

[21]Occupations that fall in the automated region include steel and metal workers, machine operators in the plastic industry, construction carpenters and continuous mining machine operators.

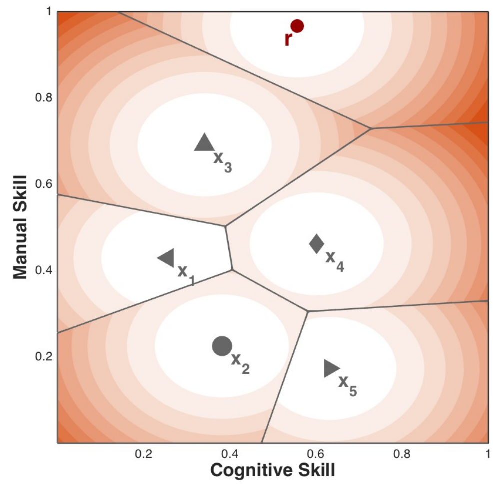
Table 2.3: Wages after Automation

	x_2	x_3	x_4	x_5
Δ%Wages	-2.0%	-12.1%	-1.5%	-0.6%

Note: The table presents the percentage change of wages for each worker type after automation. x_1 workers do not have changes in their wage.



(a) Mismatch without Automation



(b) Mismatch with Automation

Figure 2.8: Mismatch and the Assignment of Tasks to Workers

Note: The figures show the level curves of mismatch across tasks under the assignment without automation (left panel) and with automation (right panel). Darker regions correspond to higher mismatch.

mismatch for workers affected by automation but reassigned to new tasks (as x_3). Both effects decrease the difference between workers' productivity at the boundaries, reducing wages. Although in general these effects can be counteracted by the increase in output is not large enough as to increase wages.

Finally, it is worth noting that the tasks being automated are not those for which the cost of automation is lowest. As was mentioned in Section 2.2.1, the automation problem balances the cost of automation with the benefits that stem from the decrease in the mismatch in the assignment. It is by decreasing mismatch between tasks' skill requirements and workers' skills that output increases. Figure 2.8 shows the level curves of the mismatch across tasks for the assignment with and without automation. The reduction of mismatch compensates for the cost of the robot along the upper end of the skill space.

2.4 Concluding Remarks

I extend the framework developed in Chapter 1 to address the consequences of worker replacing technologies like automation or offshoring. These technologies replace workers in some, but not all, of the tasks they perform, transforming occupations.

The model makes it possible to ask about the optimal direction of automation, i.e., which type of tasks should be automated. I use data on occupations and automatability for the U.S. economy. The model rationalizes observed trends in the automation of manual intensive tasks. These tasks are automated despite there being cheaper tasks to automate because of the gains in out induces by reducing the mismatch between workers' skills and tasks' skill requirements.

Chapter 3

Efficiency Gains from Wealth Taxation

3.1 Introduction

This chapters studies how the origins of inequality affect the nature of optimal government policies. We ask a simple question: How does taxing the income flow from capital ("capital income tax") differ from taxing the stock of wealth ("wealth tax")?[1] To fix ideas, let a denote wealth, r denote the rate of return on wealth, and τ_k and τ_a denote the flat tax rates on capital income and wealth, respectively. Under a capital income tax, the after-tax wealth of individual i is given by

$$a^i_{\text{after-tax}} = a^i + (1 - \tau_k) \times ra^i,$$

whereas under the wealth tax, it is

$$a^i_{\text{after-tax}} = (1 - \tau_a) \times a^i + (1 - \tau_a) \times ra^i.$$

In a variety of benchmark economic models, the answer to the question we posed above is not very interesting: the two tax systems are equivalent, with $\tau_a = \frac{r\tau_k}{1+r}$. Partly due to this equivalence, the academic literature on capital taxes most often focuses on capital income taxes, with the understanding that they can be reinterpreted as wealth taxes. However, the equivalence result relies on the assumption that all individuals face the same rate of return on wealth, which we also made implicitly above by not indexing r with a superscript

[1] We use capital and wealth interchangeably throughout the paper.

i. What happens if this assumption does not hold—that is, if rates of return vary across individuals as some recent empirical evidence (we review below) shows?

To see some of the implications for taxation, consider the following stark but illustrative example. Two entrepreneurs start out with the same wealth level—say $1,000 each—but earn different returns on their wealth, say, $r^1 = 0$ and $r^2 = 20\%$. Under capital income taxation, the unproductive (first) entrepreneur will escape taxation because he generates no income, and the tax burden will fall entirely on the more productive (second) entrepreneur because he generates positive capital income. Under wealth taxation, on the other hand, both entrepreneurs will pay the same amount of tax regardless of their productivity, which will expand the tax base, shift the tax burden toward the unproductive entrepreneur, and reduce (potential) tax distortions on the productive entrepreneur.[2] To the extent that these differences in productivity are persistent, a wealth tax will gradually prune the wealth of idle entrepreneurs and boost that of successful ones, leading to a more efficient allocation of the aggregate capital stock, in turn raising productivity and output. In this sense, wealth taxation has a "use it or lose it" effect that is not present in capital income taxation.

While this is a very stylized example, it illustrates how (rate of) return heterogeneity can activate interesting new mechanisms that drive a wedge between the implications of the two ways of taxing capital. The main contribution of this paper is to study these implications in a full-blown quantitative overlapping-generations model with return heterogeneity. As we elaborate in a moment, we find that the two taxes have very different–sometimes opposite–implications.

There are two more considerations that motivate us to take return heterogeneity seriously for studying capital taxation. First, a growing number of new empirical studies cast strong doubt on the assumption of homogenous returns across households. Using recently available administrative panel data sets that track millions of individuals over long periods of time, these studies document large and persistent differences in rates of return across individuals, even after adjusting for risk and other factors (e.g., Fagereng et al. (2016), Bach et al. (2018), and Smith et al. (2017)).[3] In our view, these new pieces of evidence make studying the tax implications of return heterogeneity more than a theoretical curiosity.

[2] Table 3.1 povides some illustrative calculations for this example.

[3] Among these, Fagereng et al. (2016) study a 20-year-long panel that covers all households in Norway and contains extensive details about their portfolios and investments during this time. They find large differences across individuals in their (risk-adjusted) rates of return averaged over 20 years. Bach et al. (2018) analyze a similar panel data set from Sweden and conclude that the main driver of wealth inequality at the top is heterogeneity in rates of return. Finally, for the United States, Smith et al. (2017) use a unique panel data set from the US Treasury Department that contains information on 10 million firms and their owners; they document persistent heterogeneity in firm profitability even after adjusting for risk and size.

Second, another growing literature—on power law models—shows that rate of return heterogeneity is a powerful modeling tool that can generate key features of inequality that have proved challenging to explain by other mechanisms.[4] This is an important benefit for the purposes of this paper: because the wealth distribution is extremely concentrated in the United States—as well as in many other countries—the bulk of the capital tax burden falls on a small fraction of wealthy households. This makes capital taxation much more about the "right tail" than taxes on consumption and labor income, which are more evenly distributed than wealth. Thus, it seems a priori important for our model not only to generate the extreme wealth concentration at the top but also to be consistent with other features that are likely to be relevant for capturing the key trade-offs faced by very wealthy individuals.

One such feature is the thick Pareto tail of the wealth distribution seen in many countries around the world (Vermeulen (2016)), which is challenging to generate by many models of inequality (even by some of those that match the share of wealth held by the top 1%) but emerges naturally in models with return heterogeneity (Benhabib, Bisin and Zhu, 2011; Benhabib, Bisin and Luo, 2017). Further, if return heterogeneity is persistent, these models also generate behavior consistent with the *dynamics* of wealth inequality over time (Gabaix, Lasry, Lions and Moll (2016) and Jones and Kim (2018)). Another important feature that determines the trade-offs faced by the wealthy is the extent to which their wealth is dynastic/inherited or self made/accumulated. In the United States, a significant fraction of the very wealthy are self made and accumulate wealth very rapidly during their lifetime. For example, about 53% of the individuals on the 2017 US Forbes 400 list were self-made billionaires, which implies a (conservative) lower bound of a 1000-fold increase in their wealth over the life cycle. In contrast, in models of inequality that rely on idiosyncratic labor income risk or discount rate heterogeneity it takes dozens of generations for extremely high wealthy individuals to emerge. A calibrated model featuring return heterogeneity can generate this pattern, as we show in this paper.

Finally, there is also an important practical motivation for studying wealth taxation—it is a tool that has long been used by governments around the world. Until the last decade or so, many of the richest OECD countries had wealth taxation (e.g., France, Germany, Spain, Italy, Netherlands, Nordic countries, among others). Although its popularity has waned significantly in recent decades, it is still being used in France, Spain, Netherlands, Switzerland, and Norway.[5] In light of this reality, studying the effects of wealth taxation

[4]See Gabaix (2009) and Benhabib and Bisin (2018) for excellent recent reviews of this literature.

[5]As of 2017. See OECD (2018) for a recent review of the use of wealth taxes across OECD countries.

(and how they differ from capital income taxation) is an important step toward providing better guidance to policy makers.

For the quantitative analysis, we study an overlapping-generations model where individuals derive utility from consumption and leisure. The key ingredient of the model is persistent heterogeneity in investment/entrepreneurial skills, which, together with incomplete financial markets that prevent free flow of funds across agents, allows some individuals to earn persistently higher returns on their wealth than others. Individuals can borrow from others in a bond market to invest in their firm over and above their own saved resources. The same bond market can also be used as a savings device, which will be optimal for individuals whose entrepreneurial skill (and hence private return) is low or have too much wealth or both.

Each individual/entrepreneur produces a differentiated intermediate good using a linear technology and individual-specific productivity levels. These intermediates are combined in a Dixit-Stiglitz aggregator by a final goods producing firm, which pins down each entrepreneur's production scale and profits. In our calibrated economy, most individuals earn the bulk of their income from wages, and only a small fraction (10–20%, depending on the exact definition) of individuals produce large enough output to be considered an entrepreneur/investor. Individuals also face idiosyncratic labor income risk, mortality risk, borrowing constraints in the bond market, and various intergenerational links (accidental bequests to offspring, correlated entrepreneurial and labor market skills, etc.), although plausible variations in these details do not change the substantive conclusions. The calibrated model is consistent key features of the U.S. wealth distribution mentioned above, including the high concentration of wealth, the Pareto right tail, the rapid wealth growth of the very wealthy, the cross-sectional dispersion in lifetime rates of return, and the amount of borrowing by US businesses. Further, the extent of capital misallocation generated in the model is in line with the U.S. data (e.g., Bils, Klenow and Ruane (2017)).

Our analysis produces three sets of results. First, we study a revenue-neutral tax reform that replaces the current U.S. tax system of capital income taxation with a flat wealth tax, keeping taxes on labor and consumption unchanged. Comparing across steady states, this reform raises average welfare significantly—equivalent to about 7–8% of consumption (per person per year) for newborn individuals in our baseline calibration. The gains come from a combination of more efficient allocation of capital and higher capital levels generated by the use-it-or-lose-it mechanism inherent in wealth taxation. Furthermore, these welfare gains are quite evenly distributed across the population because productivity improvements raise output and wages, benefitting workers across the board.

Second, we move to an optimal tax analysis, in which a utilitarian government chooses linear taxes on labor income and on wealth to maximize the ex ante expected lifetime utility of a newborn. We repeat the same analysis, this time having the government choose linear taxes on labor and capital income. In the first case, we find that the optimal wealth tax rate is positive and relatively high, at about 3%. This allows the government to reduce the tax on labor income (from 22.5% down to 14.5%), which is more distorting than the wealth tax in this environment. The lower labor income tax reinforces the rise in before-tax wages, in turn boosting labor supply and further raising output and welfare. That said, most of the welfare gain still comes from the reduced misallocation of capital (as in the tax reform), a smaller part from higher labor supply, and almost none of it from a change in the capital stock level—which remains almost unchanged—in the new steady state. In other words, the benefits of optimal wealth taxes in this experiment do not require more capital accumulation at the aggregate level.[6]

Turning to optimal capital income taxation, the optimal tax rate turns out to be negative and large: about –35%, implying a large subsidy to capital income. At first blush, this finding may seem surprising in light of earlier results in the literature, which found a high positive tax rate (of about +35%) using lifecycle models with incomplete markets that share many similarities to ours (c.f. Conesa, Kitao and Krueger (2009)). The main difference turns out to be return heterogeneity: shutting down return heterogeneity and recalibrating our model restores the high positive tax rate found in previous work. The intuition for the capital income subsidy being optimal is relatively simple: in the standard model, the wealthy are individuals who earned high labor income in the past but are not better at investing this wealth than others. With return heterogeneity, the wealthy are primarily those who are good investors, so the redistributional benefits from taxing their income is easily outweighed by the loss from distorting their savings and reducing their wealth. This result shows that two similar models of wealth inequality (versions of the same model with and without return heterogeneity) may have very different implications—not only for wealth taxation but also for capital income taxation.

Third, we find that among the two optimal tax systems, the one with wealth taxes yields higher welfare gains (9.5%) than the one with capital income taxes (6.5%). A decomposition analysis shows that the gains under wealth taxes come from both a rise in the *level* of consumption (driven by higher after-tax wages) and a decline in the *inequality* of consumption. Thus, optimal wealth taxes yield both first- and second-order gains. This is not the

[6]Based on this result, we cautiously conjecture that an experiment that incorporates transition analysis may not result in large losses along the transition path as there may not be a need for higher savings to achieve the new steady state.

case with optimal capital income taxes: although they deliver an even larger rise in output, providing capital subsidies requires higher taxes on labor income, resulting in only a small rise in *after-tax* wages and therefore in consumption. Furthermore, subsidies on capital lead to a significant rise in inequality—both in wealth but also more importantly in consumption—yielding distributional losses, which offsets some of the gains from levels—unlike under optimal wealth taxes.

Finally, we conduct various sensitivity checks to gauge the robustness of these conclusions. In particular, we have considered progressive labor income taxes, optimal wealth taxes with an exemption level, introducing estate taxation, relaxing or eliminating borrowing constraints, different assumptions about the stochastic process governing entrepreneurial productivity, and various changes in key parameters. While these changes affect the various magnitudes of welfare gains (as would be expected), they do not overturn any of the main substantive conclusions of our analysis.

The rest of the paper is organized as follows. Section 3.2 elaborates on the simple static example described above. Section 3.3 lays out the full-blown model, and Section 3.4 describes the parameterization and model fit. Sections 3.5 and 3.6 present the quantitative results from the tax reform and optimal taxation, respectively. Section 3.7 discusses sensitivity analyses; Section 3.8 concludes.

Related Literature

Although the "use-it-or-lose-it" feature of wealth taxes has been discussed by some authors, we are not aware of prior academic work studying its effects as we do in this paper. Maurice Allais was probably one of the best-known proponents of wealth taxes who spelled out the use-it-or-lose-it rationale in his book on wealth taxation.[7] More recently, Piketty (2014) has revived the debate on wealth taxation and proposed using a combination of capital income and wealth taxes to balance these efficiency and inequality tradeoffs. Piketty mostly focused on equity considerations, but also described the use-it-or-lose-it mechanism without providing a formal analysis.[8]

The broader literature on capital taxation is vast, so we will not attempt to review it here. For thorough recent surveys, see, Chari and Kehoe (1999) and Golosov et al. (2006). This paper is more closely related to the strand that conducts quantitative analyses of

[7]He observed that "[a] tax on the capital stock represents a bonus to production and penalizes the inefficient owner, passive, for whom income taxes encourage inaction (Allais, 1977, p. 501, translated)."

[8]The work of Shourideh (2013) shares some similarities to ours. He provides a theoretical analysis of the Mirleesian taxation problem of wealthy individuals who face a risk-return trade-off in their investment choice. He finds a progressive saving tax to be the optimal policy.

capital taxation when financial markets are incomplete, tax instruments are restricted (in plausible ways), and/or individuals are finitely lived (Hubbard, Judd, Hall and Summers (1986), Aiyagari (1995), Imrohoroglu (1998), Erosa and Gervais (2002), Garriga (2003), Conesa et al. (2009), Kitao (2010)). Some of these studies found that the optimal capital tax rate may be positive and large. The two main differences of our analysis are (i) the presence of heterogeneous returns and (ii) considering wealth taxation.[9] On capital income taxation, our contribution is to show that the presence of return heterogeneity can alter the substantive conclusions and turn the optimal policy from a tax to a subsidy when the heterogeneity is sufficiently large. On wealth taxation, we show that its effects can be qualitatively very different from taxing capital income and yield larger and more broad-based welfare gains.

As noted above, this paper is also related to the literature on power law models of inequality.[10] This literature points out that the well-established thick Pareto tail of the wealth distribution is difficult to explain in Bewley-Aiyagari style models where wealth inequality is due to precautionary savings in response to idiosyncratic income shocks. This is because the wealth distribution inherits the Pareto tail of the income distribution (as shown by Benhabib et al. (2017) theoretically and by Hubmer et al. (2017) via simulations), which is significantly thinner than the tail for wealth in the data. Furthermore, when the idiosyncratic income process is estimated to match micro evidence on income dynamics, these models are able to generate plausible implications for the bottom 95% or so of the wealth distribution but severely miss inequality at the top (e.g., generate 1/3 of the wealth holdings for the top 1% and fail to generate individuals with more than $20 million in wealth, among others; see De Nardi et al. (2016), Guvenen et al. (2016), and Carroll et al. (2017).

The power law literature identifies various plausible mechanisms—such as birth/death processes, creative destruction, stochastic discount factors, and heterogeneity in returns (or growth rates)—that can give rise to a Pareto tail in the steady state distribution of wealth or income (Benhabib et al. (2011, 2013, 2014)). Furthermore, as Gabaix et al. (2016) show, when the heterogeneity in returns rates is persistent, these models generate behavior also consistent with the *dynamics* of inequality. Our model shares similarities to Jones and Kim

[9]An exception is Kitao (2008) who studies the differences between taxing capital income from entrepreneurial activities (namely profits) and capital income from rents (namely bonds). She does this in an occupational choice model where entrepreneurs differ in their productivity.

[10]We discuss here the most recent strand that focuses on inequality in income and wealth. Earlier important contributions include Gabaix (1999) on Zipf's law in city size distribution, Gabaix (2011) on whether idiosyncratic shocks to firms can cause aggregate fluctuations, Luttmer (2007, 2011) on the dynamics of firm growth and the Pareto tail in firm size distribution, as well as the much earlier literature in the 1950s that these papers build upon and extend. See Gabaix (2009); Benhabib and Bisin (2018) for detailed surveys.

(2018), who emphasize the creative destruction process in entrepreneurial production to explain the Pareto tail of the income distribution. Despite the rapid growth in this literature, the implications of capital taxation in these models have been unexplored, and our paper fills this gap.

Finally, our paper has some useful points of contact with several papers that feature (entrepreneurial) firms with heterogeneous productivity facing financial frictions, leading to misallocation of capital, lower productivity, underdevelopment, among others. Examples include Restuccia and Rogerson (2008) and Hsieh and Klenow (2009) in the context of aggregate TFP; Buera, Kaboski and Shin (2011), Moll (2014), and Itskhoki and Moll (2019) in the context of economic development; Quadrini (2000) and Cagetti and De Nardi (2009) in the context of entrepreneurship, among others. These papers do not study tax policy in general, and none of them study differences between capital income and wealth taxes as we do in this paper. One recent exception is Itskhoki and Moll (2019) who study optimal dynamic Ramsey policies in such an environment but focus on labor tax policies and capital subsidies.

3.2 A Simple One-Period Example

It is useful to elaborate on the simple example described in the Introduction. Consider two brothers, named Fredo and Michael, who each has \$1000 of wealth at time zero. Fredo has low entrepreneurial skills, and so he earns a return of $r_F = 0\%$ on his investments, whereas Michael is a highly skilled business man, and so he earns a return of $r_M = 20\%$. Both brothers invest all their wealth in their business and make no other decisions (such as consumption or saving choice). To introduce taxation, suppose that there is a government that needs to finance an expenditure of $G = \$50$ through tax revenues collected at the end of the period. The example is summarized in Table 3.1.

Now, suppose that the government taxes capital income at a flat rate. To raise \$50, the required tax rate is 25% on income and is paid entirely by Michael, who is the skilled entrepreneur and the only one earning any capital income. Consequently, the after-tax return is 0% for Fredo and 15% for Michael. By the end of the period, Fredo's wealth remained unchanged, whereas Michael experienced an increase from \$1,000 to \$1,150 after paying his taxes.

Next, suppose that the government decided to raise the same revenue with a wealth tax. Now the base of taxation is broader, because Fredo does have wealth and cannot avoid taxation as he did under the capital income tax. Specifically, the tax base covers the entire wealth stock, or \$2200, at the end of the period. The tax rate on wealth is $\$50/\$2,200 \approx$

2.27%. More importantly, Fredo's tax bill is now \$23, up from zero, whereas Michael's tax bill is cut by almost half, from \$50 before down to \$27. The after-tax rate of return is, respectively, $(\$0 - \$23)/\$1000 \approx -2.3\%$ for Fredo and $(\$200 - \$27)/\$1000 \approx 17.3\%$ for Michael. Notice that the dispersion in after-tax returns is higher under wealth taxes and the end-of-period wealth inequality is also higher: $\$1,173/\$977 \approx 1.20$ versus $\$1,150/\$1,000 = 1.15$ before. Most crucially, the more productive entrepreneur (Michael), ends up with a larger fraction of aggregate wealth: 54.6% vs. 53.5% under capital income taxes.

To sum up, wealth taxation has two main effects that are opposite to capital income taxes. First, by shifting some of the tax burden to the less productive entrepreneur, it allows the more productive one to keep more of his wealth, thereby reallocating the aggregate capital stock towards the more productive agent. Second, wealth taxes do not compress the after-tax return distribution nearly as much as capital income taxes do, which effectively punish the successful entrepreneur and reward the inefficient one. In a (more realistic) dynamic setting, such as the one we study in the next section, this feature will yield an endogenous response in savings rates, further increasing the reallocation of capital to the more productive agent, leading to a rise in productivity and output. At the same time, this reallocation process also increases wealth concentration, which may conflict with distributional goals of the society. So, overall, relative to the capital income tax, the wealth tax generates efficiency gains but can lead to distributional losses. As we shall see in the quantitative analysis, however, distributional losses are not a robust feature of wealth taxes and are mitigated or reversed (into gains) when a proper production function is introduced and wage income is added to the model. In that case, wealth taxes yield both efficiency and distributional gains.

Before we conclude this example, an important remark is in order. If this one-period example were to be repeated for many periods, all aggregate wealth—both in the capital income tax and the wealth tax cases—will eventually be owned by the more productive investor, Michael. As it turns out, as long as there are variations in the rates of return, the main arguments in favor of a wealth tax, highlighted in the simple model, remain valid. Variations in the rates of return are realistic features of the data: both over the life cycle (the fortunes of entrepreneurs do fluctuate over time) and from one generation to the next (the entrepreneurial ability of children often differs from that of their parents). Thus, we incorporate these features in the rich dynamic model we consider next.

Table 3.1: Capital Income Tax vs. Wealth Tax

	Capital Income Tax		Wealth Tax	
	$r_F = 0\%$	$r_M = 20\%$	$r_F = 0\%$	$r_M = 20\%$
Wealth	\$1,000	\$1,000	\$1,000	\$,1000
Pre-tax income	\$0	\$200	\$0	\$200
Tax rate	$\tau_k = \frac{\$50}{\$200} = 0.25$		$\tau_a = \frac{\$50}{\$2{,}200} = 2.27\%$	
Tax liability	\$0	\$50	$\$1,000 \times \tau_a \approx \$23$	$\$1,200 \times \tau_a \approx \$27$
After-tax rate of return	0%	$\frac{\$200-\$50}{\$1{,}000} = 15\%$	$-\frac{\$23}{\$1{,}000} = -2.3\%$	$\frac{\$200-\$27}{\$1{,}000} = 17.3\%$
After-tax wealth ratio	$\frac{W_M}{W_F} = \frac{\$1{,}150}{\$1{,}000} = 1.15$		$\frac{W_M}{W_F} = \frac{\$1{,}173}{\$977} = 1.20$	

3.3 Full OLG Model

We study an economy populated by overlapping generations of finitely-lived individuals, two sectors (producing intermediate-goods and the final good, respectively), and a government that raises revenues through various taxes.

3.3.1 Individuals

Individuals face mortality risk and can live up to a maximum of H years. Let ϕ_h be the unconditional probability of survival up to age h and let $s_h \equiv \phi_h/\phi_{h-1}$ be the conditional probability of surviving from age $h-1$ to h. When an individual dies, she is replaced by an offspring that inherits her wealth.

Individuals derive utility from consumption, c, and leisure, ℓ, and maximize expected lifetime utility without any bequests motives:

$$\mathbb{E}_0 \left(\sum_{h=1}^{H} \beta^{h-1} \phi_h u(c_h, \ell_h) \right).$$

Individuals make four decisions every period: (i) leisure time vs. labor supply to the market (until retirement age, $R < H$), (ii) consumption today vs saving for tomorrow, (iii) portfolio choice: how much of his own assets/wealth to invest in his own business versus how much to lend to others in the bond market, and (iv) how much to produce (of an intermediate good) as an entrepreneur. We now describe the endowments of various skills, production, technologies, and the market arrangements, and then spell out each of the four decisions in more detail.

3.3.2 Skill Endowments and their Evolution

Each individual is endowed with two types of skill: one that determines his productivity in entrepreneurial activities and another that determines his productivity as a worker. We now describe these two skills, how they evolve across generations and over the life cycle, and how they enter the two activities undertaken by the individuals.

Entrepreneurial productivity

Let z_{ih} denote the entrepreneurial productivity of individual i at age h, which has two components: $\overline{z}_i$, which is fixed over the life cycle but changes across generations (inherited from the parent), and a second component that varies stochastically over the life cycle. Specifically, a newborn inherits $\overline{z}_i$ imperfectly from her parent:

$$\log(\overline{z}_i^{\text{child}}) = \rho_z \log(\overline{z}_i^{\text{parent}}) + \varepsilon_{\overline{z}_i},$$

where $\varepsilon_{\overline{z}_i}$ is an i.i.d. normal innovation with mean zero and variance $\sigma^2_{\varepsilon_{\overline{z}}}$. Because $\overline{z}_i$ is imperfectly inherited, some children with low entrepreneurial skills will inherit large amounts of wealth from their successful parent, and vice versa, causing misallocation of productive resources.

Whereas $\overline{z}_i$ captures an individual's more permanent traits, we also want to allow for the fact that these entrepreneurial skills can be augmented with external factors (such as a lucky head-start on a new idea, good health and energy that can allow skills to be fully utilized) or hampered again by factors (such as competitors entering the field, opportunity cost of time rising due to family factors, negative health shocks, among others). To allow for these variations, we allow the individual to be in different "phases" of productivity, modeled as a three-state Markov chain that can take on the values high, low, and zero: $\mathbb{I}_{ih} \in \{\mathcal{H}, \mathcal{L}, 0\}$ at h. Together with $\overline{z}_i$, this determines the entrepreneurial productivity of an individual at a given age:

$$z_{ih} = f(\overline{z}_i, \mathbb{I}_{ih}) = \begin{cases} (\overline{z}_i)^{\lambda} & \text{if } \mathbb{I}_{ih} = \mathcal{H} \qquad \text{where } \lambda > 1 \\ \overline{z}_i & \text{if } \mathbb{I}_{ih} = \mathcal{L} \\ 0 & \text{if } \mathbb{I}_{ih} = 0 \end{cases}$$

and transition between these states is governed by the transition matrix:

$$\Pi_z = \begin{bmatrix} 1-p_1-p_2 & p_1 & p_2 \\ 0 & 1-p_2 & p_2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Finally, individuals whose permanent ability is above the median permanent ability—i.e., $\overline{z} > \overline{z}_{med} = 1$—start life in state $\mathbb{I}_{ih} = \mathcal{H}$ while the rest start in state $\mathbb{I}_{ih} = \mathcal{L}$. Overall, this structure is intended to capture the fact that many individuals who are extremely wealthy go through a very high growth phase especially, in the early stages of their business, followed by a slowdown as their business matures or their competitors catch up.

Labor market productivity

At a given age individuals differ in their labor market productivity, y_{ih}, which consists of three components

$$\log y_{ih} = \underbrace{\theta_i}_{\text{permanent}} + \underbrace{\kappa_h}_{\text{lifecycle}} + \underbrace{e_{ih}}_{\text{AR(1)}}$$

where θ_i is an individual fixed effect, κ_h is a life-cycle component that is common to all individuals and e_{ih} follows an AR(1) process during working years ($h < R$):

$$e_{ih} = \rho_e e_{i,h-1} + \epsilon_e,$$

where ϵ_e is an i.i.d. shock with mean zero and variance $\sigma^2_{\epsilon_e}$. Individual-specific labor market ability θ is imperfectly inherited from parents:

$$\theta^{child} = \rho_\theta \theta^{parent} + \epsilon_\theta,$$

where ϵ_θ is an i.i.d. Gaussian shock with mean zero and variance $\sigma^2_{\epsilon_\theta}$.

Let $n_{ih} = 1 - \ell_{ih}$ denote the labor hours supplied in the market. Individuals supply their labor services to the final goods producer, so they make up the aggregate labor supply,

$$L = \int (y_{ih} n_{ih})\, didh, \tag{3.1}$$

used in the aggregate production function (3.2) described in a moment. Therefore, for a given market wage rate per efficiency units of labor, w, an individual's labor income is given by $wy_{ih}n_{ih}$.

3.3.3 Production Technology

Final Goods Producer

The final good, Y, is produced according to a Cobb-Douglas technology,

$$\mathcal{Y} = Q^{\alpha} L^{1-\alpha}, \tag{3.2}$$

where L is the aggregate labor input defined in (3.1), and Q is the CES composite of intermediate inputs, x_i:[11]

$$Q = \left(\int x_{ih}^{\mu} didh \right)^{1/\mu}. \tag{3.3}$$

Each x_i is produced by a different individual in a way that will be specified in a moment. The final goods producing sector is competitive, so the profit maximization problem is:

$$\max_{\{x_{ih}\},L} \left(\int x_{ih}^{\mu} didh \right)^{\alpha/\mu} L^{1-\alpha} - \int p_{ih} x_{ih} didh - wL,$$

where p_i is the price of the intermediate good i. The first order optimality conditions yield the inverse demand (price) function for each intermediate input and the wage rate:

$$p\left(x_{ih}\right) = \alpha x_{ih}^{\mu-1} Q^{\alpha-\mu} L^{1-\alpha} \qquad w = (1-\alpha) Q^{\alpha} L^{-\alpha}. \tag{3.4}$$

Intermediate Goods Producers

There is a continuum of intermediate goods, each produced by a different individual according to a linear technology:

$$x_{ih} = z_{ih} k_{ih} \tag{3.5}$$

where k_{ih} is the final good (consumption/capital) used in production by entrepreneur i and z_{ih} is her stochastic and idiosyncratic entrepreneurial productivity at age h.

3.3.4 Markets and the Government

Financial markets

There is a bond market where intra-period borrowing and lending takes place at a risk-free rate of r. Individuals with sufficiently high entrepreneurial productivity relative to their

[11] To distinguish Q from the unadjusted capital stock $K := \int k_{ih} didh$, we will often refer to the former as the "quality-adjusted capital stock" since its level depends on the allocation of the capital stock across entrepreneurs (and reflects the extent of misallocation).

private assets may choose to borrow in this market to finance their business. Similarly, those with low productivity relative to their assets may find it optimal to lend for a risk-free return. Following a large literature, we impose borrowing constraints to capture information frictions or commitment problems, which we do not model explicitly (among others, Cagetti and De Nardi (2006) and Buera et al. (2011)). In particular, an individual with asset level a faces a financial constraint

$$k \leq \vartheta(z_{ih}) \times a,$$

where $\vartheta(z_{ih}) \in [1, \infty]$. The (potential) dependence of ϑ on z_{ih} is to allow for the fact that more productive agents could potentially borrow more against their personal assets.[12] When $\vartheta = 1$, the financial constraint is extreme, since individuals can only use their own assets in production. When $\vartheta = \infty$ there is no longer a financial constraint since there is no longer a restriction on the amount that an individual can borrow. We explore this last case in Section 3.7, where we show that even without misallocation of capital in the economy there is scope for efficiency and welfare gains from changing to wealth taxes. The reason for this result is the effect on capital accumulation of higher after-tax returns under wealth taxes.

Tax Systems

In the benchmark economy that aims to represent the current U.S. tax system, the government is assumed to impose flat taxes at rate τ_c on consumption (expenditures), τ_l on labor income, and τ_k on capital income. In the tax experiment we consider, we will study a revenue-neutral switch to an alternative system where the government will replace taxes on capital income (i.e., set $\tau_k \equiv 0$) with flat taxes on individuals' end of period wealth stock, τ_a, leaving labor and consumption tax rates intact. In the robustness analysis, we will consider various forms of progressivity in taxes (especially on labor income and on wealth).

The government taxes to finance social security pension payments to the retirees in the economy and an exogenously given level of government spending G.

[12]We allow for this possibility to capture the idea that the market could (perhaps partially) observe individuals' productivity level and know they are able to produce a lot and pay back their debt. We model this feature as a possibly realistic aspect of financial markets that mitigates the constraints on investment and the extent of misallocation, thereby reducing the role of wealth taxes that we study later. Li (2016) finds evidence of this relation between borrowing constraints and productivity for young, unlisted firms in Japan. With homogenous constraints, $\vartheta(z_{ih}) = \overline{\vartheta}$, the impact of wealth taxes are larger.

Social Security Pension System

When an individual retires at age R, she starts receiving social security income $y^R(\theta,e)$ that depends on her type θ in the following way:

$$y^R(\theta,e) = \Phi(\theta,e)\,\overline{E}.$$

$\overline{E}$ corresponds to the average earnings of the working population in the economy, and Φ is the agent's replacement ratio, a function that depends on the agent's permanent type θ and the last transitory shock to labor productivity. The replacement ratio is progressive and satisfies:

$$\Phi(\theta,e) = \begin{cases} 0.9\frac{y_1^R(\theta,e)}{\overline{y}_1^R} & \text{if } \frac{y_1^R(\theta,e)}{\overline{y}_1^R} \leq 0.3 \\ 0.27 + 0.32\left(\frac{y_1^R(\theta,e)}{\overline{y}_1^R} - 0.3\right) & \text{if } 0.3 < \frac{y_1^R(\theta,e)}{\overline{y}_1^R} \leq 2 \\ 0.91 + 0.15\left(\frac{y_1^R(\theta,e)}{\overline{y}_1^R} - 2\right) & \text{if } 2 < \frac{y_1^R(\theta,e)}{\overline{y}_1^R} \leq 4.1 \\ 1.1 & \text{if } 4.1 < \frac{y_1^R(\theta,e)}{\overline{y}_1^R} \end{cases}$$

where $y_1^R(\theta,e)$ is the average efficiency units over lifetime that an agent of type θ gets conditional on having a given $e_R = e$.

$$y_1^R(\theta,e_R) = \frac{1}{R}\int_{h<R,a,\mathbf{S}} y_h(\theta,e)\,d\Gamma(h,a,\mathbf{S}).$$

$\mathbf{S} = (\overline{z},\mathbb{I},\theta,e)$ is the vector of exogenous states of an individual, and the integral is taken with respect to the stationary distribution (Γ) of agents such that e_R is the one given in the left hand side. Finally $\overline{y}_1^R$ is the average of $y_1^R(\theta,e)$ across θ and e.

For future reference let SSP denote the aggregate value of "social security pension" payments:

$$SSP := \int_{h\geq R,a,\mathbf{S}} y^R(\theta,e)\,d\Gamma(h,a,\mathbf{S}).$$

3.3.5 Individual's problem

For clarity of notation, in this subsection we suppress the individual subscript i. The production problem of each individual is static: funds for investment are borrowed z is observed in a period and are repaid at the end of the period. So, this problem can be solved in isolation of her other decisions.

Individual/Entrepreneur's Problem

First, as an entrepreneur, the individual chooses the optimal capital level to maximize profit:

$$\begin{aligned} \pi\left(a,z\right) &= \max_{k\leq\vartheta(z)a}\left\{p\left(zk\right)\times zk-\left(r+\delta\right)k\right\} \\ &\text{s.t. } p\left(zk\right)=\mathcal{R}\times\left(zk\right)^{\mu-1}, \end{aligned} \tag{3.6}$$

where δ is the depreciation rate of capital, $z=f(\overline{z}_i,\mathbb{I}_{ih})$, and $\mathcal{R}=\alpha Q^{\alpha-\mu}L^{1-\alpha}$, which yields the solution:

$$k\left(a,z\right)=\min\left\{\left(\frac{\mu\mathcal{R}z^{\mu}}{r+\delta}\right)^{\frac{1}{1-\mu}},\vartheta(z)a\right\}. \tag{3.7}$$

Then, the maximized profit function is:

$$\pi\left(a,z\right)=\begin{cases}\mathcal{R}\left(z\vartheta(z)a\right)^{\mu}-\left(r+\delta\right)\vartheta(z)a & \text{if } k\left(a,z\right)=\vartheta(z)a \\ \left(1-\mu\right)\mathcal{R}z^{\mu}\left(\frac{\mu\mathcal{R}z^{\mu}}{r+\delta}\right)^{\frac{\mu}{1-\mu}} & \text{if } k\left(a,z\right)<\vartheta(z)a\end{cases}. \tag{3.8}$$

The after-tax non-labor income, $Y(a,z,\tau_k,\tau_a)$, is given by after-tax profits from their firm and interest payments obtained from the financial market:

$$Y(a,z,\tau_k,\tau_a)=\left[a+\left(\pi\left(a,z\right)+ra\right)\left(1-\tau_k\right)\right]\left(1-\tau_a\right). \tag{3.9}$$

Individual's Dynamic Programming Problem

The individual's problem then is given by:

$$\begin{aligned} V_h(a,\mathbf{S}) &= \max_{c,n,a'} u\left(c,1-n\right)+\beta s_{h+1}E\left[V_{h+1}(a',\mathbf{S}')\mid\mathbf{S}\right] \\ &\text{s.t. } \left(1+\tau_c\right)c+a'=Y\left(a,z,\tau_k,\tau_a\right)+y_h^W\left(\theta,e\right) \\ &\quad a'\geq 0, \end{aligned}$$

where

$$y_h^W\left(\theta,e\right)=\begin{cases}(1-\tau_l)wy_hn & \text{if } h<R \qquad \text{where } \log y_h=\theta+\kappa_h+e \\ y^R(\theta,e) & \text{if } h\geq R.\end{cases}$$

We assume that $e_h=e_{h-1}$ for $h\geq R$, thus the retirement income is essentially conditioned on the earnings shock in period $R-1$.

3.3.6 Equilibrium

Let $c_h(a, \mathbf{S})$, $n_h(a, \mathbf{S})$ and $a_{h+1}(a, \mathbf{S})$ denote the optimal decision rules and $\Gamma(h, a, \mathbf{S})$ be the stationary distribution of agents. A competitive equilibrium is given by the following conditions:

1. Consumers maximize given $p(x)$, w, r and taxes.
2. The solution to the final goods producer gives pricing function $p(x)$ and wage rate w.
3. $Q = \left(\int_{h,a,\mathbf{S}} (z \times k(a, z))^{\mu} \, d\Gamma(h, a, \mathbf{S})\right)^{1/\mu}$ and $L = \int_{h,a,\mathbf{S}} (y_h(\theta, e)\, n_h(a, \mathbf{S}))\, d\Gamma(h, a, \mathbf{S})$, where $\log y_h = \theta + \kappa_h + e$.
4. The government budget balances.

$$\begin{aligned} G + SSP &= \tau_k \int_{h,a,\mathbf{S}} (\pi(a, z) + ra)\, d\Gamma(h, a, \mathbf{S}) \\ &+ \tau_a \int_{h,a,\mathbf{S}} (\pi(a, z) + (1 + r)\, a)\, d\Gamma(h, a, \mathbf{S}) \\ &+ \tau_L \int_{h,a,\mathbf{S}} (w y_h(\theta, e)\, n_h(a, \mathbf{S}))\, d\Gamma(h, a, \mathbf{S}) \\ &+ \tau_c \int_{h,a,\mathbf{S}} c_h(a, \mathbf{S}) d\Gamma(h, a, \mathbf{S}) \end{aligned}$$

   where

$$SSP = \int_{h \geq R,a,\mathbf{S}} y^R(\theta, e)\, d\Gamma(h, a, \mathbf{S}).$$

   We will compare the the equilibrium of the economy under capital income taxes ($\tau_k \neq 0$, $\tau_a = 0$) and under wealth taxes ($\tau_k = 0$, $\tau_a \neq 0$).
5. The bond market clears:

$$0 = \int_{h,a,\mathbf{S}} (a - k(a, z))\, d\Gamma(h, a, \mathbf{S})$$

3.4 Quantitative Analysis

3.4.1 Model Parameterization

The benchmark model is calibrated to the U.S. data. The model period is one year.

Government policy. The current U.S. tax system is modeled as a triplet of tax rates: on capital income (τ_k), labor income (τ_l), and consumption expenditures (τ_c). Following McDaniel (2007) who measures these tax rates for the U.S. economy, we set the capital income tax rate to $\tau_k = 25\%$, the labor income tax rate to $\tau_\ell = 22.4\%$, and the consumption tax rate to $\tau_c = 7.5\%$.

Demographics. Individuals enter the economy at age 20 and can live up to age 100 (i.e., a maximum of 81 periods). They retire at age 64 (model age $R = 45$). The conditional survival probabilities from age h to $h+1$ are taken from Bell and Miller (2002) for the U.S. data.

Preferences. In the baseline analysis, we consider a Cobb-Douglas utility function:

$$u(c, \ell) = \frac{(c^\gamma \ell^{1-\gamma})^{1-\sigma}}{1-\sigma}.$$

We set $\sigma = 4$ following Conesa et al. (2009). We then choose γ and β (the subjective time discount factor) to generate an average of 40 hours of market work per week for the working-age population (i.e., $\ell = 0.6$, assuming 100 hours of discretionary time per week) and a wealth-to-output ratio of 3, which requires $\gamma = 0.46$ and $\beta = 0.9475$.

Labor market efficiency. The deterministic life-cycle profile, κ_h, is modeled as a quadratic polynomial that generates a 50% rise in average labor income from age 21 to age 51.[13] The annual persistence of the autoregressive process for labor income, ρ_e, is set to 0.9.[14] The standard deviation of the innovation, σ_e, is set to 0.2. The intergenerational correlation of the fixed effect of labor market efficiency, ρ_θ, is set to 0.5, which is broadly consistent with the estimates in the literature (see Solon (1999) for a survey). Finally, with these parameters fixed, we set $\sigma_{\epsilon_\theta} = 0.305$ so as to match our empirical target of a cross-sectional standard deviation of log labor earnings of 0.80 (Guvenen, Karahan, Ozkan and Song, 2015a).

Entrepreneurial productivity. The evolution of entrepreneurial ability across generations is governed by the parameters ρ_z and $\sigma_{\varepsilon_{\overline{z}}}$. Unfortunately, there is not much empirical evidence on either parameter from the U.S. data that we are aware of. In light of this, we turn to evidence from other countries. In particular, Fagereng et al. (2016) estimate individual fixed effects in rates of return over a 20-year period for parents and their children from administrative panel data on Norwegian households. They report a small correlation of about 0.1, which we take as our empirical value of ρ_z. We also conducted robustness analysis using a value of $\rho_z = 0.5$ but did not find any substantive differences. As for, $\sigma_{\varepsilon_{\overline{z}}}$

[13] $\kappa_{ih} = \frac{60(h-1)-(h-1)^2}{1800}$

[14] See Guvenen (2007) and others.

we choose it so as to match the share of aggregate wealth held by the top 1% of the wealth distribution.

In calibrating the stochastic component of entrepreneurial ability, one concern we have in mind is the inability of many models of wealth inequality to generate the speed at which the super wealthy—or the self-made billionaires—emerge in the data. In contrast, in these models the extreme wealth concentration emerges at a very slow pace and often requires hundreds of years. Thus, one target we match is the fraction of self-made billionaires in the Forbes 400 list. The classification adopted by Forbes is shown in Table C.1 in the appendix. We define a self-made billionaire to be one who came from an upper-middle-class or lower-income family (Categories 8–10 in Table C.1). By this definition 54% of individuals on the list are self made. The model counterpart is defined as an individual who inherits less than one million dollars and goes on to become a billionaire. We set $\lambda = 5$, $p_1 = 0.05$, and $p_2 = 0.03$, which generates a self-made ratio of billionaires of 50%.

Production. We target a labor share of output of 0.60 by setting $\alpha = 0.4$. The curvature parameter of the CES aggregator of intermediate inputs, μ, is set to 0.9. With this value, our model generates the Pareto tail of the wealth distribution as it is observed in the U.S. data (see Figure 3.1). Later, we will provide robustness checks on its value. The depreciation rate of capital is set to 5%.

Financial constraint. We allow firms with higher productivity to borrow more. In particular, we choose

$$\vartheta(\bar{z}_i) = 1 + 1.5(i-1)/8 \text{ for } i = 1, ..., 9.$$

Note that we have 9 grid points for the permanent component of z. Table 3.2 summarizes the parameters that we calibrate independently (top panel) and those that are calibrated jointly (bottom panel) in equilibrium to match the moments shown in Table 3.3.

3.4.2 Performance of the benchmark model

The parametrized model is consistent with the facts on wealth inequality, the distribution of average lifetime returns on investment, the level of non-financial business liability, and the overall level of misallocation in the economy. As reported in Table 3.4, the model also matches several other moments that are not targeted in the calibration. First, the model generates bequest-to-wealth ratio that is broadly consistent with the data despite all bequests being accidental in the model. Second, tax revenues as a fraction of GDP and the capital tax share of total tax revenues the model generates are close to their counterparts

Table 3.2: Benchmark Parameters

Parameters Calibrated Outside of the Model		
Parameter		Value
Capital income tax rate	τ_k	0.25
Labor income tax rate	τ_l	0.224
Consumption tax rate	τ_c	0.075
Exponent of labor tax function (baseline)	ψ	0.00
Wealth tax rate (baseline)	τ_a	0.00
Autocorrelation for idiosyncratic labor efficiency	ρ_e	0.9
Std. for idiosyncratic labor efficiency	σ_{ϵ_e}	0.2
Interg. correlation of labor fixed effect	ρ_θ	0.5
Intermediate goods aggregate share in production	α	0.4
Curvature parameter of CES production func.	μ	0.9
Depreciation rate	δ	0.05
Curvature of utility function	σ	4.0
Maximum age	H	81
Retirement age	R	45
Survival probabilities	ϕ_h	Bell and Miller (2002)
Parameters Calibrated Jointly in Equilibrium		
Discount factor	β	0.9475
Consumption share in utility	γ	0.460
Std. dev. of interg. transmission of entrepreneurial ability	$\sigma_{\varepsilon_{\bar{z}}}$	0.072
Std. dev. of interg. transmission of labor fixed effect	σ_{ϵ_θ}	0.305
Productivity boost	λ	5.0

Table 3.3: Targeted Moments

	U.S. Data	Benchmark
Top 1%	0.36	0.36
Wealth-to-output ratio	3.00	3.00
Std. dev. of log earnings	0.80	0.80
Average Hours	0.40	0.40
Fraction self made	54%	50%

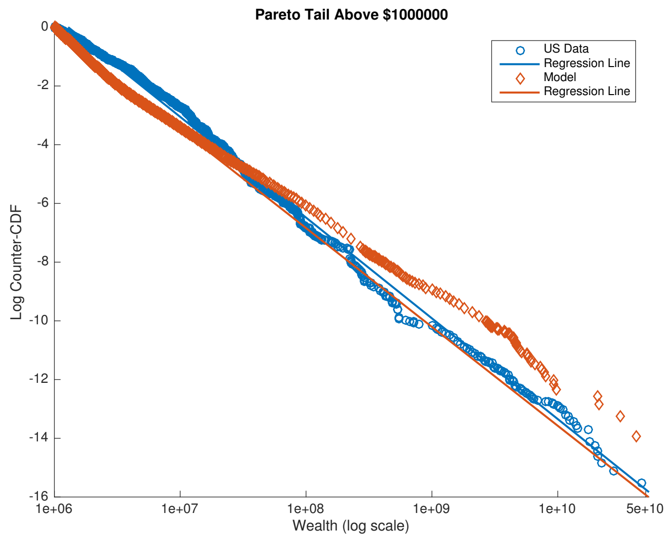
Table 3.4: Statistics of the Benchmark Model

	U.S. Data	Benchmark
Bequest/Wealth	1–2%	0.99%
GDP share of total tax revenue	0.295	0.25
Revenue share of capital tax	0.280	0.25
GDP share of capital tax	0.083	0.063
Mean return on wealth	6.9	8.33
GDP share of aggregate debt	1.29	1.27

in the data. The fact that the model is consistent with the data along these key dimensions implies that this framework is appropriate for studying capital—either capital income or wealth—taxation.

Wealth Inequality. Figure 3.1 plots the Pareto tail, for those with a wealth level higher than one million, from the benchmark calibration and the one reported in Vermeulen (2016) who merges Survey of Consumer Finance (SCF) and Forbes 400 data in 2010. The model generates a clear Pareto tail of the wealth distribution that is remarkably similar to the one in the U.S. data. As discussed earlier, most models fail to generate a realistic Pareto tail of the wealth distribution; however, power law models with rate-of-return heterogeneity are successful in generating a realistic Pareto tail. The model also matches the concentration of wealth: the top 1% of wealthiest individuals hold 36% of the wealth in the economy. At the top of the wealth distribution, as reported in Table 3.5 and Figure 3.1, the concentration of wealth is slightly higher in the model than in the data – the top 0.1% of wealthiest individuals hold 14% of the wealth in the U.S. and 23% in the benchmark model. The shape of the Pareto tail is closely linked to the curvature parameter μ, which determines the degree to which returns fall as an individual becomes richer (or, to be more precise, the capital employed in his business grows). In the robustness analysis, we have experimented with different values of μ and found that the Pareto shape is preserved for values of μ higher than 0.8 while for lower values it becomes concave. Finally, note that we targeted the fraction of billionaires that are self-made and importantly the model is consistent with the data along that dimension.

Figure 3.1: Pareto Tail - Wealth above 1 Million



Note: The Pareto tail is computed for agents with wealth of at least one million dollars. U.S. data is taken from Vermeulen (2016) who merges SCF and Forbes 400 data for 2010.

Table 3.5: Wealth Concentration in the Benchmark Model

	U.S. Data	Benchmark
Top 0.1%	0.14	0.23
Top 0.5%	0.27	0.31
Top 1%	0.36	0.36
Top 10%	0.75	0.66
Top 50%	0.99	0.97
Wealth Gini	0.82	0.78

Note: Wealth shares are computed using data for the U.S. from Vermeulen (2016) who merges SCF and Forbes 400 data for 2010. The wealth Gini is computed from 2001 SCF and is taken from Wolff (2006).

Table 3.6: Deviation of percentiles of the distribution of lifetime returns relative to the median

	p99.9	p99	p90	p75	p25	p10
Norwegian Data	19.9%	9.7%	4.1%	2.1%	-1.3%	-2.4%
Working life	19.4%	13.3%	7.8%	4.5%	-2.9%	-3.7%
Ages 20-24	55.4%	32.7%	13.3%	4.9%	-5.6%	-10.0%
Ages 25-65	19.9%	13.6%	8.0%	4.7%	-2.9%	-3.4%

Note: Lifetime returns are weighted by the individual's wealth at each age. All numbers are before tax. All numbers are presented as differences from the median. The Norwegian data is taken from Fagereng et al. (2016), Table 4, which reports percentiles of fixed effects of individual returns to wealth.

Lifetime returns in the benchmark model. The heterogeneity in the rates of returns is an important mechanism in the model for generating a wealth distribution that is consistent with the data in numerous dimensions. Therefore, it is of interest to compare the dispersion in the rates of return in the model and in the data. Even though, the empirical evidence is scarce, Fagereng et al. (2016) report the rates of returns in the Norwegian data. Rather encouragingly, the dispersion observed in the model matches well with the facts reported in Fagereng et al. (2016).

The return at age h for individual i is given by:

$$\text{Return}_{ih} = 100\frac{ra_{ih} + \pi\left(a_{ih}, z_{ih}\right)}{a_{ih}}$$

where π is defined as in equation (3.6). The lifetime return for individual i is computed as the weighted average over the individual's working life, weighted by the individual's wealth at each age:

$$\text{Return}_{i} = \sum_{h=1}^{R} \frac{a_{ih}}{\sum_{h=1}^{R} a_{ih}} \text{Return}_{ih}.$$

Table 3.6 reports various percentiles in the lifetime rates of return distribution in the data and in the model, relative to the median return in the data and in the model, respectively. The lifetime rate of return at the 99.9th percentile, relative to the median return, is around 20% both in the model and in the data. The lifetime returns at other percentiles above the median, however, are slightly higher than the returns observed in the data–e.g., the lifetime return at the 99th percentile is around 10% in the data and around 13-14% in the model. As expected, the rates of return are substantially higher at high percentiles when individuals are young. As productive individuals experience significant growth early in the life cycle, between the ages of 20 and 24, they experience rates of return as high as 55%

at the 99th percentile. Overall, the distribution of lifetime rates of returns in the model is consistent with the distribution observed in the Norwegian data.[15]

Non-financial business liability to GDP ratio. The Federal Reserve Statistical Release (2015Q3) reports that in the first quarter of 2015, the total nonfinancial business liability in the United States was $22.79 trillion compared to the nominal GDP for that quarter of $17.65 trillion, which implies an aggregate debt-to-GDP ratio of 1.29.[16] Asker, Farre-Mensa and Ljungqvist (2011) report an average debt-to-asset ratio of 0.20 for publicly-listed firms and a ratio of 0.31 for private firms in the United States. Given that the capital-to-output ratio is 3 in our model, these numbers correspond to an aggregate debt-to-output ratio of between 0.60 to 0.93. The aggregate debt-to-GDP ratio in the model is the same as in the Federal Reserve Statistical Release (2015Q3) and slightly higher than the numbers reported in Asker, Farre-Mensa and Ljungqvist (2011). The severity of the financial constraint is a quantitatively important aspect of the analysis. The fact that the financial constraint is, if anything, lower in the model than in the data indicates that in the benchmark analysis we will not be overstating the extent of capital misallocation: a tighter financial constraint, which will results in a debt-to-GDP ratio similar to the level of leverage reported in Asker, Farre-Mensa and Ljungqvist (2011), will yield even higher welfare gains from wealth taxation in the analysis that follows.

Misallocation in the benchmark model. Our benchmark economy is distorted due to the existence of financial frictions in the form of borrowing constraints, and we can measure the effects of these distortions on aggregate TFP and output and compare them to those obtained in other studies. A large and growing literature frames the discussion on misallocation in terms of various wedges, such as capital, labor, and output wedges. The analysis in Hsieh and Klenow (2009) is particularly useful since, in a similar model environment, they study the degree of misallocation and its effect on TFP in manufacturing in China, India, and the United States. Hsieh and Klenow (2009) use detailed firm-level data from the U.S. Census of Manufacturers (1977, 1082, 1987, 1992, and 1997) and find that the TFP gains from removing all distortions (wedges), which equalizes the "Revenue Productivity" (TFPR) within each industry, is 36% in 1977, 31% in 1987, and 43% in 1997.

[15]The overall message remains unchanged if we instead compute the lifetime rates of return percentiles in the model relative to the median return in the Norwegian data.

[16]See line 19 of Table L.102 of the Flow of Funds Z1 Integrated Macroeconomic Accounts in Federal Reserve Statistical Release (2015Q3). In a previous version of this draft, we used the figure on credit market borrowing by Nonfinancial Sectors, Table L2, line 18 (page 10) from Federal Reserve Statistical Release (2015Q1)). The figure used to be $12.2 trillion implying a ratio of 0.68.

We can follow the approach of Hsieh and Klenow and compute the same measures of misallocation for the U.S. as in their analysis. Instead of modeling and capturing the effect of a particular distortion, or distortions, the approach in Hsieh and Klenow (2009), and the related misallocation literature, is to infer the underlying distortions and wedges in the economy by studying the extent to which the marginal revenue products of capital and labor differ across firms in the economy (or in a particular industry). This is based on the insight that absent any distortions, the marginal revenue products of capital and labor have to be equalized across all firms.[17]

Appendix C.2 provides the details as to how we map our model into the wedge analysis environment in Hsieh and Klenow (2009). Their analysis measures the improvement in total output as a result of an improvement in TFP in all industries. In our model, this corresponds to the improvement in TFP in the Q sector. We find that removing the capital wedges would increase total output, through its effect on TFP in the Q sector, by 20%—this is approximately half of the gains reported by Hsieh and Klenow. However, in ongoing research Bils et al. (2017) propose a method for correcting measurement error in micro data and find that TFP gains from removing distortions in the U.S. are rather in the range of 20%, very much in line with the results from our benchmark economy. Therefore, we conclude that the level of distortion in our model environment is not far from the actual amount of distortion present in the U.S. economy.

3.5 Tax Reform: Replacing Capital Income Tax with Wealth Tax

Our first major experiment is to study the effects of a tax reform in which the government eliminates capital income taxes (setting $\tau_k = 0$) from the baseline economy, keeps τ_l and τ_c unchanged, and levies a flat-rate wealth tax so as to keep the tax revenue fixed at its level in the baseline economy.

An important detail in the analysis, however, is the fact that pension benefits, as described in Section 3.3.4, are a function of the average labor income in the economy, and thus any change in the level of income implies a change in the level of aggregate social security payments and hence would lead to an unbalanced budget if revenue is kept constant. To deal with this issue, we consider two cases. In the first case, which is our main "revenue

[17] This is the case in the monopolistic competition models, such as in Hsieh and Klenow (2009). Alternatively, in environments such as in Lucas (1978) and Restuccia and Rogerson (2008), in which firms feature decreasing returns to scale, but produce the same homogeneous good, in the non-distorted economy the marginal products of capital and labor have to be equalized.

neutral" tax reform experiment, we keep the pension income of every individual fixed at its baseline value after the wealth tax reform. In the second case, the "balanced budget" tax reform experiment, we allow pension benefits to scale up or down with the level of average labor income in the economy, while choosing the level of wealth taxes to keep the government budget balanced. Except where we note explicitly, all results we discuss pertain to the first case – the revenue neutral tax reform.

3.5.1 After-tax Returns and Reallocation of Wealth

Variable changes. Table 3.7 lists the values of the aggregate variables in the baseline economy and their percentage change after the wealth tax reform. A revenue-neutral tax reform requires that the capital income tax of 25% be replaced with a wealth tax of 1.13%. If the tax reforms allows for pension benefits to be indexed to the average labor income in the economy, then a slightly higher wealth tax of 1.54% is required.
All variables of interest increase substantially. Aggregate capital increases by 19.4% with the tax reform. Moreover, Q (effective or quality-adjusted capital) increases even more, by 24.8%. The larger increase in Q relative to $\bar{k}$ reflects the fact that wealth is more concentrated in the hands of more productive agents under the wealth tax, reflecting the efficiency gains associated with the wealth tax. The increase in Q drives up other aggregate variables as well. The aggregate output increases by 10.1%, labor supply increases by 1.3%, and the wage rate increases by 8.7%. The general equilibrium increase in the wage rate is critical in distributing more evenly the welfare gains from the tax reform to the whole population since labor efficiency is more evenly distributed than wealth.
Table 3.8 shows some key statistics on wealth in the benchmark and the tax-reform economies. The wealth distribution becomes more concentrated at the top under the wealth tax: the share of wealth held by the top 1% increases from 36% to 46%, while the fraction held by the top 10% increases from 66% to 72%. The wealth-to-output ratio also increases from 3.0 to 3.25.

Mechanisms at play. The simple one-period example in Section 3.2 provided the main insights for the effects of a change from a capital income tax to a wealth tax. Most importantly, productive entrepreneurs face a higher after-tax rate of return on their investment under the wealth tax than under the capital income tax while the opposite was true for low productive entrepreneurs. The same key mechanism is at play in the much richer benchmark environment. To illustrate that, Table 3.9 shows various percentiles of the after-tax return distribution. Indeed, the after-tax rates of return at the 99th (10th) percentile are

Table 3.7: Tax Reform: Macro Variables in the Baseline Economy and After Reform

		Benchmark	Tax Reform	
			τ_a	τ_a + SS
Capital income tax rate	τ_k	25%	0.0	0.0
Wealth tax rate	τ_a	0	1.13%	1.54%
		Level	(Δ% from benchmark)	
Aggregate capital	$\bar{k}$	3.50	19.4	12.3
Intermediate goods	Q	3.51	24.8	18.4
Wage	w	1.25	8.7	6.4
Output	$\mathcal{Y}$	1.17	10.1	7.9
Labor	L	0.56	1.3	1.4
Consumption	C	0.83	10.0	8.4

Note: The last column labeled "$\tau_a + SS$" reports the results from the "balanced budget" experiment in which pensions payments are allowed to change as average labor income changes with the tax reform.

Table 3.8: Key Variables: Benchmark Calibration vs. Tax Reform

	Data	Benchmark	Tax Reform
Top 1%	0.36	0.36	0.46
Top 10%	0.75	0.66	0.72
Wealth/Output	3.00	3.00	3.25
Average hours	0.40	0.40	0.41
Std of log earnings	0.80	0.80	0.80
Bequest/Wealth	1–2%	0.99	1.07

Table 3.9: Changes in the Return Distribution

	P10	P50	P90	P95	P99
	Before-tax				
Benchmark	2.00	2.00	17.28	22.35	42.36
Wealth Tax	1.74	1.74	14.62	19.04	36.91
	After-tax				
Benchmark	1.50	1.50	12.96	16.76	31.77
Wealth Tax	0.59	0.59	13.32	17.69	35.35

Note: Each cell reports the rate of return in percentages.

around 35.4% (0.6%) under the wealth tax and 31.7% (1.5%) under the capital income tax – after-tax returns increase at upper percentiles and decrease at lower percentiles of the return distribution. This increase in the dispersion in after-tax returns mechanically increases the concentration of wealth.

Overall, there is substantial reallocation of wealth towards more productive agents when the capital income tax is replaced with a wealth tax. Table 3.10 reports, for a particular top x% of the wealth distribution, the percentage change in the fraction of agents with a particular entrepreneurial productivity. For example, among the top 1% in the wealth distribution, the fraction of individuals in the top 10% of the productivity distribution increased at the expense of less productive agents, resulting in a reallocation of wealth towards more productive entrepreneurs. This increased aggregate efficiency is reflected in higher quality-adjusted capital, Q, and results in higher output, consumption, and wages, as we already pointed out.

3.5.2 Welfare Analysis

In order to quantify the welfare consequences of the tax reform, we use the following two measures.

CE_1 : This measure is constructed at the individual level and then aggregated up. In particular, we first compute the consumption equivalent welfare for each individual in a particular state and then integrate it over the population, using the stationary distribution in the benchmark economy:[18]

[18] Given our utility function specification, the welfare consequences of switching from the benchmark economy to a counterfactual economy with a wealth tax for an individual in state $\mathbf{S}$ with age h and wealth a is given by

Table 3.10: Tax Reform from τ_k to τ_a: Change in Wealth Composition

	Productivity group (Percentile)						
Top x%	0-40	40-80	80-90	90-99	99-99.9	99.9-99.99	99.99+
1	-12.0	-13.0	-10.8	10.5	11.2	9.8	6.9
5	-8.2	-3.3	1.6	8.3	8.9	8.1	6.2
10	-6.4	-1.3	2.9	6.4	6.9	6.3	5.0
50	-2.5	0.9	1.8	1.6	1.2	1.1	1.1

Note: The table shows the percentage change induced by the tax reform from τ_k to τ_a of the share of agents in each entrepreneurial productivity group (ranked based on the permanent component of entrepreneurial productivity $\overline{z}$) among the top x% wealth holders (i.e. agents above the x^{th} percentile of the wealth distribution). Each entry is computed as $100 \times \frac{s_{ij}(\tau_a)-s_{ij}(\tau_k)}{s_{ij}(\tau_k)}$, where i indexes groups of top x% wealth holders, j indexes entrepreneurial productivity groups and τ the tax regime.

Table 3.11: Average Welfare Gains from Tax Reform

	Baseline		Baseline + SS reform	
	$\overline{CE}_1$	$\overline{CE}_2$	$\overline{CE}_1$	$\overline{CE}_2$
Average CE for newborns	7.40%	7.86%	5.58%	4.71%
Average CE	3.14%	5.14%	4.95%	4.10%
% in favor of reform	67.8% .		94.8%	

$$V_0((1+CE_1(\mathbf{s}))c^*_{\mathrm{US}}(\mathbf{s}), \ell^*_{\mathrm{US}}(\mathbf{s})) = \mathbb{V}_0(c(\mathbf{s}), \ell(\mathbf{s}))$$

$$\overline{CE}_1 \equiv \sum_{\mathbf{s}} \Gamma_{\mathrm{US}}(\mathbf{s}) \times CE(\mathbf{s}),$$

where $\mathbf{s} = (a, h, \mathbf{S})$ and V_0 and $\mathbb{V}_0$ are the lifetime value functions in the benchmark (U.S.) capital income tax economy and the counterfactual wealth tax economy, respectively. This measure allows us to discuss individual-specific outcomes and to understand "who gains, and who loses, and by how much" from the tax reform.

$$CE_h(a, \mathbf{S}) = 100 \times \left[\left(\frac{V_h(a, \mathbf{S}; \tau^{policy})}{V_h(a, \mathbf{S}; \tau^{bench})} \right)^{1/\gamma(1-\sigma)} - 1 \right].$$

This measure specifically gives what fraction of consumption an individual is willing to pay in order to move from the steady state of the economy with a capital income tax to the steady state of the economy with a wealth tax.

CE_2 : The second measure is simpler, and more similar to the famous Lucas (1987) calculation: it measures the fixed proportional consumption transfer to all individuals in the benchmark economy so that the average utility is equal to that in the tax-reform economy:

$$\sum_{\mathbf{s}} \Gamma_{\mathrm{US}}(\mathbf{s}) \times V_0((1+\overline{CE}_2)c^*_{\mathrm{US}}(\mathbf{s}), \ell^*_{\mathrm{US}}(\mathbf{s})) = \sum_{\mathbf{s}_0} \Gamma(\mathbf{s}) \times \mathbb{V}_0(c(\mathbf{s}), \ell(\mathbf{s})).$$

Findings. Table 3.11 provides the overall welfare gains from switching from a capital income to a wealth tax. The welfare gains are large: 3.14% for the whole population using the CE_1 measure and 5.14% using the CE_2 measure. The average welfare gain for newborn individuals is even higher: 7.40% and 7.86%, respectively, for the two different welfare measures. Overall, 68% of all individuals across the whole population in the benchmark economy prefer to be in an economy with a wealth tax.

Table 3.12 illustrates the extent to which individuals in different parts of the state distribution gain from the tax reform, based on the CE_1 welfare measure. Panel A reports the results when the pension benefits are not adjusted for changes in the average labor income in the economy. Young individuals, at the age of 20, experience substantial welfare gains, and these gains increase with productivity. Those at the top of the productivity distribution experience the largest welfare gains – they are able to grow faster and get higher after-tax returns under the wealth tax than under the capital income tax. Young individuals at the bottom of the productivity distribution also experience substantial welfare gains even though they hold very little wealth – those gains are due to higher wages in the wealth-tax economy.

The welfare gains decline with age for all levels of productivity and even become negative for individuals over the age of 65. Low productive agents do save for precautionary reasons and for retirement and imposing a wealth tax instead of a capital income tax late in life results in lower after-tax returns and is costly for them. Older high productivity entrepreneurs, on the other hand, experience low welfare gains, and even welfare losses, since some of them have lost their productivity and the wealth tax is costlier for them than the capital income tax. Retirees mostly lose from the reform since their benefits are fixed at the benchmark level, and they mostly face lower after-tax return on their savings under the wealth-tax economy. These considerations are reflected in the observed support for the reform from various part of the age-productivity distribution, as reported in Panel A in Table 3.13.

Panel B in Table 3.12 reports the welfare gains in the case when the pension benefits are adjusted for changes in the average labor income in the economy and the wealth tax is chosen in order to keep the government budget balanced. The main difference of note is the

Table 3.12: Welfare Gain by Age Group and Entrepreneurial Ability

(a) Baseline Tax Reform

	Productivity group (Percentile)						
Age	0-40	40-80	80-90	90-99	99-99.9	99.9-99.99	99.99+
20	7.0	7.3	7.9	8.9	10.6	11.6	12.4
21–34	6.5	6.3	6.3	6.6	7.0	6.9	5.7
35–49	5.1	4.4	3.9	3.3	1.7	0.4	-2.2
50–64	2.3	1.8	1.4	0.8	-0.6	-1.7	-3.5
65+	-0.2	-0.3	-0.4	-0.6	-1.2	-1.7	-2.7

(b) Tax Reform with Social Security Reform

	Productivity group (Percentile)						
Age	0-40	40-80	80-90	90-99	99-99.9	99.9-99.99	99.99+
20	4.9	5.3	6.0	7.2	9.3	10.4	11.4
21–34	4.7	4.6	4.8	5.4	6.1	6.3	5.2
35–49	4.2	3.7	3.4	2.8	1.4	0.0	-2.8
50–64	4.9	4.3	4.0	3.2	1.4	0.0	-2.3
65+	7.2	6.7	6.4	5.8	4.3	3.2	1.2

Note: Each entry reports the average welfare gain (CE_1) from the tax reform from τ_k to τ_a of agents in a given age and entrepreneurial productivity group (ranked based on the permanent component of entrepreneurial productivity $\bar{z}$). The average is computed with respect to the benchmark distribution.

Table 3.13: Fraction with Positive Welfare Gain by Age and Entrepreneurial Ability

(a) Baseline Tax Reform

	Productivity group (Percentile)						
Age	0-40	40-80	80-90	90-99	99-99.9	99.9-99.99	99.99+
20	96.1	95.8	97.2	98.0	98.7	98.9	99.0
21–34	97.3	96.3	95.8	95.0	92.6	89.9	82.5
35–49	95.8	92.7	89.5	83.9	70.7	60.7	43.7
50–64	79.4	74.5	70.2	62.9	51.1	44.1	34.4
65+	8.0	9.5	9.5	8.8	7.3	6.2	4.8

(b) Tax Reform with Social Security Reform

	Productivity group (Percentile)						
Age	0-40	40-80	80-90	90-99	99-99.9	99.9-99.99	99.99+
20	94.3	94.6	95.9	97.3	98.6	98.9	99.0
21–34	95.9	94.7	94.4	94.0	91.7	89.1	82.0
35–49	95.4	92.3	89.5	84.2	71.4	61.4	44.4
50–64	96.6	93.7	90.7	83.7	70.1	61.1	48.5
65+	99.5	98.6	97.5	92.8	82.0	73.9	60.3

Note: Each entry reports the share of agents in a given age and entrepreneurial productivity group (ranked based on the permanent component of entrepreneurial productivity $\bar{z}$) that would experience a positive welfare gain (CE_1) from the tax reform from τ_k to τ_a. The shares are computed with respect to the benchmark distribution.

fact that individuals over the age of 65 now experience welfare gains rather than welfare losses. They are benefiting from the higher efficiency in the economy under the wealth tax since their pensions reflect the higher average labor income in the economy. This results in larger support for the reform from those groups of the population, as reported in Panel B in Table 3.13.

3.6 Optimal Taxation

The discussion so far illustrates that a wealth tax is a better way of taxing capital than a capital income tax. A natural question, however, is whether taxing capital in this framework would be a part of the optimal tax schedule to begin with, and, if so, whether it is better to do it through capital income or wealth taxes. We study quantitatively this question by performing two experiments: (i) we find the optimal taxes in an environment where the government uses proportional labor income taxes and proportional capital income taxes, and (ii) we find the optimal taxes in an environment where the government uses proportional labor income taxes and proportional wealth taxes.

3.6.1 Main Results

An overview. Table 3.14 summarizes the main results: the optimal capital income tax is negative at -34.4% with a corresponding labor income tax of 36% while the optimal wealth tax is positive at 3.06% with a corresponding labor income tax of 14.1%. The optimal wealth tax delivers the highest welfare gain, 9.61%, while under the optimal capital income tax the welfare gain, 6.28%, is even lower than the 7.86% welfare gain in the tax reform experiment.

Table 3.15 shows the percentage change in aggregate variables relative to their benchmark levels once the optimal taxes are implemented. As can be seen from the table, the optimal capital income tax leads to much larger increases in output and wages. However, after-tax wages increase significantly more under the optimal wealth tax.

Figure 3.2 provides a more comprehensive picture of the optimal taxation results. It illustrates the average welfare gain of a newborn, using the CE_2 measure, relative to the benchmark, as we vary the taxes on capital or wealth. The red line corresponds to the welfare gain in the capital income tax economy and the blue line corresponds to the one in the wealth tax economy. The x-axis corresponds to the tax revenue from capital as a fraction of total tax revenue. Note that total tax revenue ($G + SSP$) is fixed in this experiment. Thus, as we vary the taxes on capital, the labor income tax adjusts to balance

Table 3.14: Optimal taxation: statistics

	τ_k	τ_ℓ	τ_a	$\frac{Thresh.}{\overline{E}}$	% Taxed	Top 1%	$\overline{CE}_2$ (%)
Benchmark	25%	22.4%	–	–	100%	0.36	–
Tax reform	–	22.4%	1.13%	0	100%	0.46	7.86
Opt. τ_k	–34.4%	36.0%	–	–	100%	0.56	6.28
Opt. τ_a	–	14.1%	3.06%	0	100%	0.47	9.61
Opt. τ_a − Threshold	–	14.2%	3.30%	25%	63%	0.48	9.83

Note: The optimal threshold amounts to 25% of the average earnings of the working population in the benchmark economy $(\overline{E})$.

the government budget. The benchmark capital income tax economy with capital income tax economy corresponds to 0.25 on the x-axis since the capital tax revenue as a fraction of total tax revenue is 0.25 in that economy.

The first observation from Figure 3.2 is that the average welfare gain of the newborn increases as the capital *income* tax is reduced below its benchmark level in the capital income tax economy so that the optimal capital income tax turns out to be –34.4%. This is in sharp contrast to the findings in the recent literature on capital income taxation, most notably Conesa et al. (2009) who find that the optimal capital income tax is 36%. We will discuss in Section 3.7 the reasons for obtaining a different policy recommendation. In the wealth-tax economy, the average welfare of the newborn increases as we increase the wealth tax, and the optimal wealth tax is positive and substantial at 3.06%. At the optimal wealth tax, the tax revenue from capital/wealth is more than 40% of the total tax revenue, which is higher than the benchmark level of 25%.

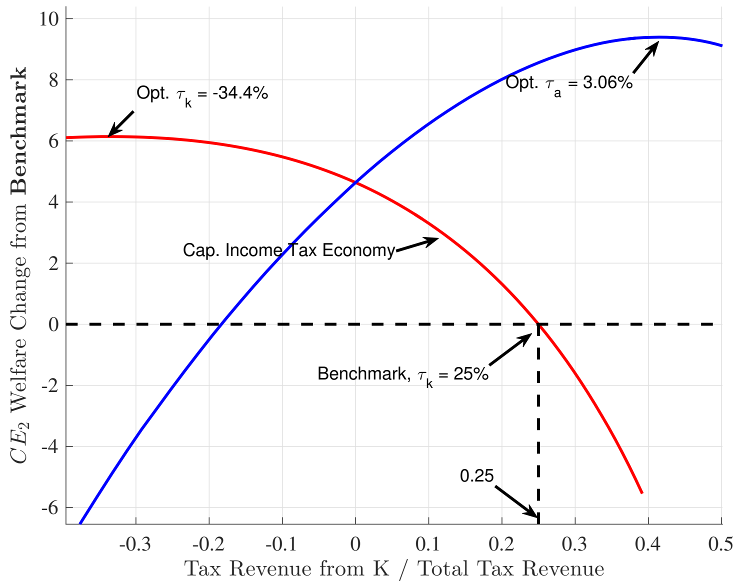
We have also studied the optimal wealth tax allowing for a threshold level below which the wealth is not taxed. In this experiment, the government maximizes welfare by choosing jointly the wealth threshold level, the wealth tax rate that applies above that threshold, and the labor income tax rate. We find that the optimal threshold level is 25% of the average earnings of the working population in the benchmark economy and the optimal wealth tax rate is 3.3%. In this case, only 63% of the population pays wealth taxes. The aggregate welfare gain from its implementation is 9.83%, which is higher than the 9.61% welfare gain from the optimal linear wealth tax. The additional aggregate welfare gain is small relative to the overall welfare gains from the implementation of the wealth tax instead of capital income tax. However, there are some important differences in distribution of welfare gains and political support for wealth taxes between a linear wealth tax system and a wealth tax

Table 3.15: Optimal Taxation: Percentage Change in Aggregate Variables

	$\%\Delta K$	$\%\Delta Q$	$\%\Delta L$	$\%\Delta Y$	$\%\Delta w$	$\%\Delta w$ (net)	Δr	Δr (net)	$\%\Delta TFP$
Tax Reform	19.37	24.79	1.28	10.10	8.70	8.70	–0.25	–0.90	4.60
Opt. τ_k	68.97	79.57	–1.16	25.51	26.97	4.72	–1.51	–0.87	6.29
Opt. τ_a	2.76	10.26	3.90	6.40	2.41	13.42	0.68	–1.92	7.29
Opt. τ_a Threshold	0.41	8.12	3.67	5.42	1.70	12.48	0.78	-2.07	7.70

Note: Percentage changes are computed with respect to the benchmark economy without wealth taxes and capital income taxes of 25%. Changes in the interest rate are computed in percentage points. The net wage is defined as $(1-\tau_l)\,w$, and the net interest rate is defined as $(1+(1-\tau_k)\,r)\,(1-\tau_a)-1$. The TFP variable is measured in the intermediate goods market.

Figure 3.2: Welfare Gain from Optimal Taxes



system with a threshold, which we report in Section 3.6.2.

Mechanisms at play. These results can be intuitively explained using the information provided in Panels A-D of Figure 3.3. As Panel A illustrates, raising taxes on capital – either through a capital income tax or a wealth tax – reduces aggregate capital $\overline{k}$ and Q. However, there are two notable differences between these two ways of raising taxes. First, aggregate capital $\overline{k}$ decreases less under the wealth tax system than under the capital income tax system. Second, Q declines more than $\overline{k}$ under the capital income tax system while it declines less than $\overline{k}$ under the wealth tax system.

We first explain the second result since it is critical for understanding the first one. Consider a simplified version of our model where before-tax gross return is given as $1 + Pz$, with z being the entrepreneurial productivity and P being the price of Q. The after-tax gross returns are then $1 + Pz\,(1 - \tau_k)$ and $(1 + Pz)\,(1 - \tau_a)$ under the capital income and wealth taxes, respectively. Consider two individuals as in our simple example, i.e. Michael and Fredo such that $z_M > 0$ and $z_F = 0$. The first observation to point out is that an increase in the capital income tax has no effect on Fredo's after tax gross return since $z_F = 0$, but reduces Michael's after-tax return, as in Section 3.2. Thus, the capital income tax mainly distorts the wealth accumulation of more productive agents, which reduces their wealth share, increases the misallocation of capital, and leads to a larger decline in Q than $\overline{k}$. With the wealth tax on the other hand, $(1 + Pz)\,(1 - \tau_a)$ is affected at the same rate for both agents for a given P. However, with a higher wealth tax, Q goes down and P

increases. Now consider these two individuals' after-tax returns: $(1 + Pz_F)(1 - \tau_a)$ versus $(1 + Pz_M)(1 - \tau_a)$. The general equilibrium increase in P partially offsets the decline in the after-tax return $(1 + Pz_M)(1 - \tau_a)$ for Michael when the wealth tax is increased. However, Fredo's after-tax return does not benefit from the increase in P since $z_F = 0$. Thus, a higher τ_a has a smaller negative impact on the more productive Michael's after-tax return. This mechanism reallocates wealth to productive agents and reduces the misallocation of capital, and leads to a smaller decline in Q than $\overline{k}$ as the wealth tax is increased.

Since the distortionary effects of capital taxes is much smaller under the wealth tax than under the capital income tax, the government can increase the wealth tax without significantly distorting output (and wages) as seen in Panel B, and can reduce the labor income tax so that the after-tax wage increases with the wealth tax. Panel C shows that the after-tax wage rate indeed increases with the wealth tax but declines with the capital income tax. Panel D illustrates that capital income is declining with capital taxes under both tax systems but it declines by less under the wealth tax system. Thus, for a given tax revenue from capital, since the after-tax wage and capital income are higher under wealth taxes, people will accumulate more assets and aggregate capital will be higher under the wealth tax.

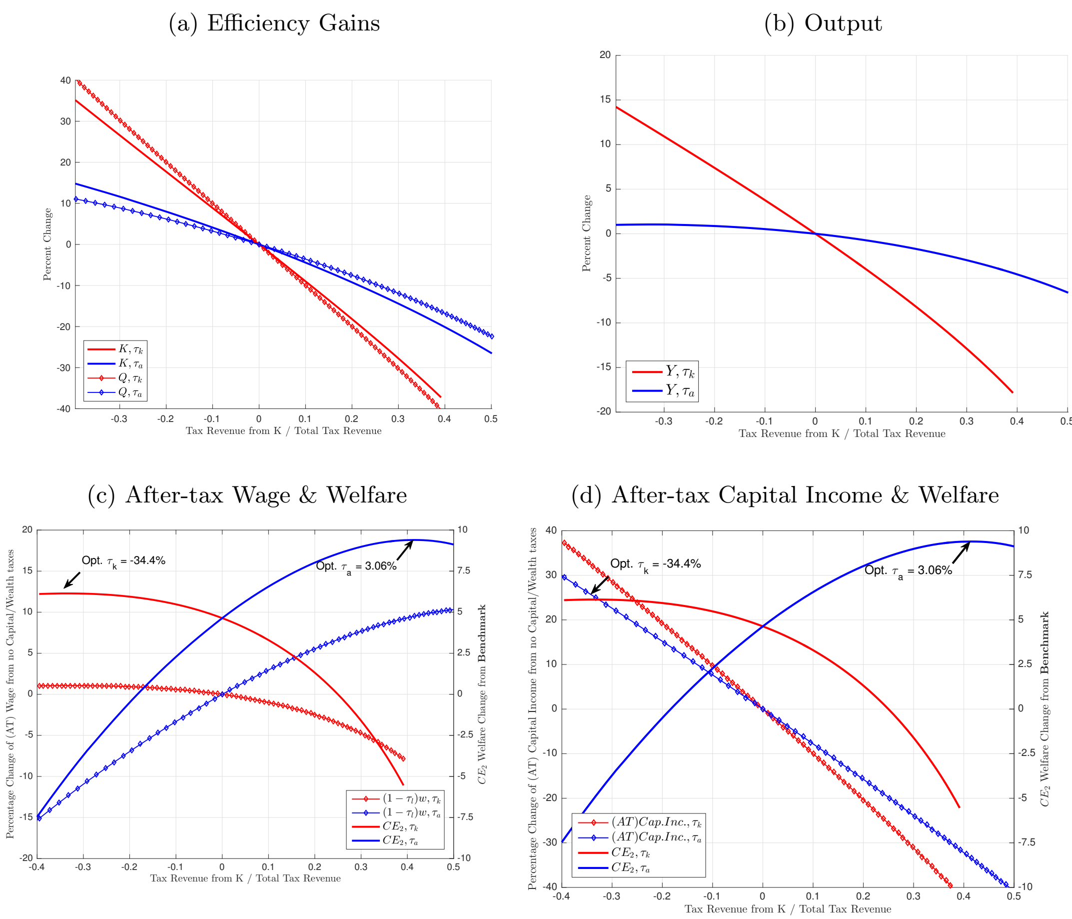
The mechanisms described above are also closely linked to the optimal tax level found under these two tax systems. Individuals whose resources mostly consist of labor income will gain from the wealth tax. Those whose resources are mainly from wealth, will lose from it. Since wealth is much more concentrated in the hands of very few agents and labor income is more evenly distributed across the population, our welfare measure, which weighs rich and poor at the same rate and maximizes the welfare of a newborn whose income is more influenced by wages, picks up a rate that is close to the rate that actually maximizes the after-tax wage rate. This point is illustrated in Panel C. Similarly, under the capital income tax economy, the after-tax wage is maximized when the capital income tax is negative. Thus, we obtain a negative optimal capital income tax.

3.6.2 Distribution of Welfare Gains and Political Support

The two panels in Table 3.16 illustrate the welfare gains, by age and entrepreneurial ability, when the benchmark capital income tax is replaced with the following two tax systems: the optimal capital income tax and the optimal (linear) wealth tax.[19] Welfare gains are typically higher for younger agents in all of these tax systems. However, there are some

[19] The results based on an optimal (linear) wealth tax with a threshold limit are similar to those in the optimal wealth tax, and we refer to them when appropriate.

Figure 3.3: Optimal Taxes on Capital



(a) Efficiency Gains

(b) Output

(c) After-tax Wage & Welfare

(d) After-tax Capital Income & Welfare

Table 3.16: Welfare Gain by Age Group and Entrepreneurial Ability

(a) Optimal **Capital Income Taxes**

	Productivity group (Percentile)						
Age	0-40	40-80	80-90	90-99	99-99.9	99.9-99.99	99.99+
20	4.0	5.6	7.2	9.5	13.0	14.8	16.1
21–34	3.7	5.0	6.2	7.9	10.4	11.4	11.2
35–49	2.7	3.3	3.8	4.0	3.5	2.7	0.7
50–64	1.1	1.4	1.6	1.5	0.6	-0.2	-1.9
65+	-0.1	0.1	0.2	0.2	-0.2	-0.7	-1.6

(b) Optimal **Wealth Taxes**

	Productivity group (Percentile)						
Age	0-40	40-80	80-90	90-99	99-99.9	99.9-99.99	99.99+
20	10.0	9.7	10.1	11.1	13.1	14.3	15.3
21–34	9.2	7.9	7.3	7.1	6.6	5.9	3.1
35–49	6.8	4.9	3.7	2.1	-1.3	-3.9	-8.8
50–64	2.7	1.4	0.6	-0.8	-3.7	-5.8	-9.3
65+	-0.6	-0.9	-1.2	-1.8	-3.2	-4.3	-6.3

(c) Optimal **Wealth Taxes - Threshold**

	Productivity group (Percentile)						
Age	0-40	40-80	80-90	90-99	99-99.9	99.9-99.99	99.99+
20	9.9	9.8	10.3	11.4	13.4	14.6	14.5
21–34	9.1	8.0	7.4	7.2	6.6	5.6	5.9
35–49	6.7	4.9	3.6	1.9	-1.6	-4.9	-4.4
50–64	2.7	1.5	0.6	-0.8	-3.9	-6.5	-6.2
65+	-0.4	-0.7	-1.0	-1.6	-3.2	-4.6	-4.4

Note: Each entry reports the average welfare gain (CE_1) from the corresponding optimal tax experiment of agents in a given age and entrepreneurial productivity group (ranked based on the permanent component of entrepreneurial productivity $\bar{z}$). The average is computed with respect to the benchmark distribution.

Table 3.17: Fraction with Positive Welfare Gain by Age Group and Entrepreneurial Ability

(a) Optimal **Capital Income Taxes**

	Productivity group (Percentile)						
Age	0-40	40-80	80-90	90-99	99-99.9	99.9-99.99	99.99+
20	95.4	98.6	99.3	99.6	99.8	99.8	100.0
21–34	96.3	97.7	97.7	97.3	96.0	94.9	92.3
35–49	91.7	92.8	91.1	87.8	80.3	74.5	63.7
50–64	74.2	76.2	73.8	69.4	60.3	53.8	43.8
65+	13.8	18.6	18.7	18.2	16.6	15.2	13.0

(b) Optimal **Wealth Taxes**

	Productivity group (Percentile)						
Age	0-40	40-80	80-90	90-99	99-99.9	99.9-99.99	99.99+
20	94.5	93.1	93.3	94.6	95.8	96.1	95.8
21–34	95.7	92.6	90.5	88.8	84.2	79.4	67.0
35–49	91.3	82.8	76.5	68.2	53.6	44.6	34.0
50–64	72.6	62.9	56.1	49.4	39.8	34.5	27.2
65+	2.1	2.3	1.8	1.4	0.9	0.7	0.4

(c) Optimal **Wealth Taxes - Threshold**

	Productivity group (Percentile)						
Age	0-40	40-80	80-90	90-99	99-99.9	99.9-99.99	99.99+
20	94.5	93.1	93.3	94.6	95.8	95.9	96.0
21–34	95.6	92.4	90.4	88.5	83.8	77.6	78.9
35–49	91.1	82.4	76.0	67.8	53.2	43.3	44.3
50–64	76.4	66.7	59.6	52.5	42.3	35.8	36.6
65+	75.9	68.6	63.7	57.9	48.7	42.1	42.9

Note: Each entry reports the share of agents in a given age and entrepreneurial productivity group (ranked based on the permanent component of entrepreneurial productivity $\bar{z}$) that would experience a positive welfare gain (CE_1) from from the corresponding optimal tax experiment. The shares are computed with respect to the benchmark distribution.

Table 3.18: Welfare Gains and Political Support

	$\overline{CE}_2$ (%)	Vote (%)
Benchmark	–	–
Tax reform	7.86	67.8
Opt. τ_k	6.28	69.7
Opt. τ_a	9.61	60.7
Opt. τ_a – Threshold	9.83	78.9

important differences. First, focusing on the working age population, we observe that the welfare gains are typically higher under wealth taxes than under capital income taxes for agents with lower entrepreneurial ability. This is directly related to the fact that after-tax wages are much higher under optimal wealth taxes than under optimal capital income taxes. Second, retirees typically experience welfare losses with the implementation of the optimal tax system under both tax systems. However, welfare losses are higher in the optimal wealth tax case. This is mainly because the after-tax interest rate is lower in this case: for example, Table 3.15 shows that the after-tax interest rate r(net) is 1.92% lower under wealth taxes than in the benchmark economy, while it is lower by only 0.87% under capital income taxes relative to the benchmark. Thus, retirees whose retirement benefits are fixed at the benchmark level and whose capital income declines due to the decline in the after-tax interest rate experience larger welfare losses when wealth taxes are implemented rather than capital income taxes. We also analyzed separately the optimal wealth tax with a threshold and found that in that case many of the low ability retirees experience lower welfare losses since they no longer pay taxes on wealth as their wealth is not that high.

Table 3.17 reports the fraction of households with positive welfare gains for each age-ability group. Red numbers correspond to less than 50% support within a group. We notice that the fraction of retirees that prefer wealth taxes is smaller than the fraction of retirees that prefer capital income taxes – that reduces the support for wealth taxes. Thus while, overall, 69.7% of the population prefers to be in the capital income tax economy, 60.7% of the population prefers to be in the wealth tax economy, and the retirees are key for understanding the larger support for capital income taxes. Once we introduce a threshold in the wealth tax, the support for the wealth tax increases among the retirees and 78.9% of the population are now in favor of the optimal wealth tax with a threshold, as shown in Table (3.18).

Table 3.19: Decomposition of Welfare Gain – CE_2 for Newborn

	Tax Reform	Opt. τ_k	Opt. τ_a
$CE_2(NB)$ (%)	7.86	6.28	9.61
		Consumption	
Total	8.27	5.90	11.02
Level	10.01	21.04	8.28
Distribution	–1.58	–12.51	2.53
		Leisure	
Total	–0.38	0.36	–1.27
Level	–0.66	0.73	–2.21
Distribution	0.27	–0.38	0.76

Decomposition of the welfare gains. Following Conesa et al. (2009), we decompose the aggregate welfare gain into a component arising from changes in consumption and a component arising from changes in leisure. Further, these changes in welfare can be decomposed into components arising from the change in average consumption (leisure) and changes in the distribution of consumption (leisure).[20][21] Table 3.19 reports these decomposition results. First, notice that the 9.61 percent welfare gain under the optimal wealth tax (τ_a) is due to an 11.02 percent welfare gain in consumption and a 1.27 percent welfare loss in leisure. Second, focusing on consumption, we observe that both an increase in the level and an improvement in the distribution positively contribute to the total welfare gain: by 8.28 percent and 2.53 percent, respectively. This is an important point worth emphasizing

[20]A similar decomposition was earlier proposed by Flodén (2001), where total welfare changes are expressed in terms of changes in levels, changes in uncertainty, and changes in inequality.

[21]Let CE be the aggregate welfare gain, and CE_C and CE_L be the components of the aggregate welfare gain arising from changes in consumption and leisure respectively. CE_C is given by

$$V_0((1+CE_C(\mathbf{s}))c^*_{\text{US}}(\mathbf{s}), \ell^*_{\text{US}}(\mathbf{s})) = \widetilde{\mathbb{V}}_0(c(\mathbf{s}), \ell^*_{\text{US}}(\mathbf{s}))$$

and CE_L is given by

$$V_0((1+CE_L(\mathbf{s}))c^*_{\text{US}}(\mathbf{s}), \ell^*_{\text{US}}(\mathbf{s})) = \widetilde{\mathbb{V}}_0(c^*_{\text{US}}(\mathbf{s}), \ell(\mathbf{s})).$$

Note that $1+CE = (1+CE_C)(1+CE_L)$. Furthermore, CE_C can be decomposed into level $CE_{\overline{C}}$ and distribution component CE_{σ_C} as

$$V_0((1+CE_{\overline{C}}(\mathbf{s}))c^*_{\text{US}}(\mathbf{s}), \ell^*_{\text{US}}(\mathbf{s})) = \widehat{\mathbb{V}}_0(\widehat{c}(\mathbf{s}), \ell^*_{\text{US}}(\mathbf{s}))$$

where $\widehat{c}(\mathbf{s}) = c^*_{\text{US}}(\mathbf{s})\frac{\overline{C}}{\overline{C}^*_{US}}$ and

$$\widehat{\mathbb{V}}_0\left((1+CE_{\sigma_C})\,\widehat{c}(\mathbf{s}), \ell^*_{\text{US}}(\mathbf{s})\right) = \widetilde{\mathbb{V}}_0(c(\mathbf{s}), \ell^*_{\text{US}}(\mathbf{s}))$$

where one can show that $1+CE_C = (1+CE_{\overline{C}})(1+CE_{\sigma_C})$. Similar decomposition applies to leisure.

– despite the fact that wealth inequality becomes much higher under the optimal wealth tax, the distribution of consumption becomes more equal relative to our benchmark, which contributes to the overall welfare gain from wealth taxes. This pattern is different from the determinants of the 5.90 percent welfare gains due to consumption under the capital income tax (τ_k)—a large 21.04 percent is due to an increase in the average level of consumption which is offset by a 12.51 percent welfare loss due to a substantial increase in consumption inequality.

3.7 Robustness

In this section, we explore the robustness of our results by conducting a sensitivity analysis with respect to the following changes in the economic environment: 1) the labor income tax is allowed to be progressive, 2) the stochastic component of entrepreneurial ability is eliminated, and we consider only permanent productivity differences, 3) the constraint on borrowing for the entrepreneur is eliminated, i.e. $\vartheta = \infty$, 4) the curvature of intermediate good production is decreased to $\mu = 0.8$, 5) estate tax is allowed, 6) wealth is measured as present value rather than book value, and 7) the rate of return heterogeneity is eliminated by setting $z_i = 1$ for all i and $\mu = 1$ (we refer to this case as "CKK" since this framework then becomes quite similar to the framework used in Conesa et al. (2009)). In all of these cases, we follow the same calibration procedure as in our benchmark economy – i.e., we target the same set of moments with the same set of parameters, except in (i) the permanent productivity type case, where we do not target the fraction of self-made billionaires, and (ii) the CKK case, where the model is unable to match the wealth concentration in the data. The results from the tax reform experiments are presented in Table 3.20, and the results from the optimal tax experiments are presented in Table 3.21. The message from all of these experiments is that our substantive quantitative conclusions are robust to any of these changes in the economic environment.

Progressive labor income tax. We introduce progressive labor income taxation, letting the after-tax labor income be defined as $(1 - \tau_l)(wy_hn)^{\psi}$, and following Heathcote, et al (2014) we set $\psi = 0.815$. τ_l is chosen so that the average labor income tax rate is 0.224 – the same as in our benchmark. In the tax reform experiment, we keep the labor income tax unchanged. As seen in Table (3.20) the results are quantitatively quite similar to those in our benchmark. In the optimal tax experiment, we search for the optimal level and progressivity of the labor income tax τ_l and ψ jointly with capital taxes. We find that the optimal progressivity of the labor income tax should be higher, which is reflected in a smaller ψ. The optimal levels of the capital income and wealth taxes are quite similar to those in the benchmark calibration.

Permanent productivity type. When we eliminate the stochastic component of entrepreneurial ability, we increase the dispersion of the permanent component in order to generate the same amount of wealth concentration. However, this version of the model can only generate 18.5% self-made richest individuals. We find that the welfare gains from the tax reform are smaller than in the benchmark, but still very large. In this version of the model there is less misallocation since more persistent productivity allows agents to self-finance and alleviate the restrictions of the borrowing constraint (see Moll (2014)). The optimal capital income tax is slightly negative at -2.33% while the optimal wealth tax is still positive and large at 2.21%.

No borrowing constraint: $\vartheta = \infty$ In this case, the marginal returns are equalized across individuals, and the misallocation of capital is completely eliminated. Yet, surprisingly, replacing the capital income tax with a wealth tax does increase welfare. Table 3.20 shows that aggregate capital $\overline{k}$ and effective capital Q increase by 6.28%. Note that they increase at the same rate since there is no misallocation. Thus, the increase in aggregate capital generates the welfare gain from switching to a wealth tax. In order to illustrate why aggregate capital increases, consider an individual's after-tax non-labor income when the financial constraint is eliminated. The entrepreneurial profit is given as

$$\pi^*(z) = \max_k \left\{ \mathcal{R} \times (zk)^\mu - (r+\delta) k \right\}.$$

The after-tax non-labor income, $Y(a, z, \tau_k, \tau_a)$, is given by

$$Y(a, z, \tau_k, \tau_a) = \begin{cases} (1 + r(1-\tau_k)) a + \pi^*(z)(1-\tau_k) & \text{under capital income tax} \\ \\ (1+r)(1-\tau_a) a + \pi^*(z)(1-\tau_a) & \text{under wealth tax.} \end{cases}$$

When the capital income tax is replaced with a wealth tax, there are two opposing mechanisms at play. We will illustrate these mechanisms for a given interest rate and distribution of agents across states. First, we can show that $(1+r)(1-\tau_a) a < (1 + r(1-\tau_k)) a$, which will reduce capital accumulation under wealth taxes. Second, $\pi^*(z)(1-\tau_a) > \pi^*(z)(1-\tau_k)$ – in fact, $\pi^*(z)(1-\tau_a)$ will be much larger than $\pi^*(z)(1-\tau_k)$ for high

z types since $\tau_k = 25\%$ and τ_a is less than 2%. The second mechanism will increase capital accumulation, especially for the most productive agents with high $\pi^*(z)$ since their after-tax profits will increase substantially.[22] Ultimately, the second mechanism dominates resulting in an increase in the aggregate capital stock when the economy switches from a capital income to a wealth tax.

Turning to the optimal tax experiment, we find that the optimal capital income tax is positive at 13.6%, but still smaller than the benchmark level of 25%. The optimal wealth tax is 1.57%, which is close to the benchmark tax reform level of 1.65%. And finally, the optimal wealth tax delivers a higher welfare gain than the optimal capital income tax.

Estate taxes. We incorporate in the model an estate tax of 40%, with an exemption level of bequests below \$5 Million, in order to capture the level of estate taxation in the U.S.. We recalibrate the benchmark economy and conduct the tax reform and optimal tax experiments holding the estate taxes fixed. The results are not much different from those in the benchmark model. The welfare gains are larger in all three experiments: tax reform, optimal capital income tax, and optimal wealth tax and all our conclusions remain unchanged.

Curvature parameter in the CES production function: $\mu = 0.80$. Holding other parameters fixed, a higher curvature (lower μ) in the CES production function implies lower efficiency gains from switching to a wealth tax since high productivity agents will face diminishing marginal productivity more quickly as they accumulate more wealth under the wealth tax. However, with a higher curvature, the model generates lower wealth concentration. Thus, recalibration requires a higher dispersion in $\bar{z}_i$ $(\sigma_{\varepsilon_{\bar{z}}})$ in order to match the wealth concentration in the top 1%. Table 3.20 shows that the welfare gain from switching to a wealth tax is very similar to the one in our benchmark. Table 3.21 shows that the welfare gain is larger for the optimal capital income tax (7.38%) and lower for the optimal wealth tax (8.32%) as compared to those in the benchmark case But still the welfare gain under the optimal wealth tax remains larger than the one under the optimal capital income tax.

[22] Note that $G = \tau_k \sum (ra + \pi^*(z)) \Gamma(a, z, :) = \tau_k (rK + \pi^*(z))$ under capital income tax and $G = \tau_a \sum ((1+r)a + \pi^*(z)) \Gamma(a, z, :) = \tau_a ((1+r)K + \pi^*(z))$ under wealth tax. Using these equations we can show that 1) $\tau_a << \tau_k$, thus $\pi^*(z)(1-\tau_a) > \pi^*(z)(1-\tau_k)$ and 2) $1 + r(1-\tau_k) = 1 + r - \frac{G}{K + \frac{\sum \pi^*(z)}{r}}$ is greater than $(1+r)(1-\tau_a) = 1 + r - \frac{G}{K + \frac{\sum \pi^*(z)}{1+r}}$.

Table 3.20: Robustness: **Tax Reform** Experiments

	Baseline	Prog. Labor Tax	Constant z	No Constr.	$\mu = 0.8$	Estate Taxes	Present Value	CKK
τ_a	1.13%	0.90%	1.23%	1.65%	1.24%	0.95%	1.26%	1.92%
				Welfare Gain from Tax Reform				
CE_1 (All)	3.14	2.79	2.29	0.44	3.07	3.56	2.47	0
CE_1 (New born)	7.40	6.48	5.46	1.86	7.54	8.22	6.07	0
CE_2 (All)	5.14	4.68	2.92	0.36	5.06	5.85	4.21	0
CE_2 (New born)	7.86	7.06	5.36	1.43	7.85	8.80	6.48	0
Vote (%)	67.7	69.0	68.3	55.9	70.2	68.4	66.9	-
				Percentage Change in Aggregates				
$\overline{k}$	19.37	21.27	9.56	6.28	16.43	21.05	15.60	0
Q	24.79	25.61	22.37	6.28	21.25	27.90	19.87	0
w	8.70	9.25	7.66	2.10	7.77	9.75	7.08	0
Y	10.10	10.01	9.54	3.02	8.38	11.25	8.18	0
L	1.28	0.69	1.75	0.91	0.57	1.37	1.04	0
C	10.01	10.01	11.25	2.93	8.33	11.31	8.17	0

Present value. We measure wealth in the model based on the book value, a, of individual's assets. This is the approach followed in the related literature as well. However, some of the wealth moments and statistics from the data could potentially be based on the market (or discounted present) value rather than book value of assets. As a result, we have experimented with a case when wealth measures in the model are based on the expected present value of future earnings from the firm, discounted by the average rate of return in the economy. In this version of the model the dispersion of wealth turns out to be higher for given set of parameter values than in the benchmark. Thus, we recalibrate the model and reduce the dispersion in $\bar{z}_i$ ($\sigma_{\varepsilon_{\bar{z}}}$) in order to match the wealth concentration in the top 1%. This recalibration reduces somewhat the welfare gains in all our experiments, although they still remain large.

Comparison to Conesa et al. (2009). One of the major differences between our model and the one studied in Conesa et al. (2009) is the rate of return heterogeneity. For comparison, we eliminate the return heterogeneity by setting $z = 1$ for all individuals and $\mu = 1$. In this case, as we have mentioned earlier, capital income taxes and wealth taxes are equivalent. Column CKK in Table 3.20 confirms this result – there are no changes in allocations nor any welfare gains from switching to a wealth tax. When we study optimal capital income taxes in this case, we find that the optimal capital income tax rate is 25.4%, which is consistent with the 36% value found in Conesa et al. (2009). This confirms their result that in an OLG model with idiosyncratic labor income risk and incomplete markets, the optimal capital income tax is positive and substantial. However, note that we find a smaller optimal capital income tax in our CKK experiment than they do. The reason for that is that accidental bequests are inherited by newborn individuals in our version of the CKK model while they are distributed equally to the whole population in their framework. Thus, in their framework, newborns start their life with less wealth and as a result prefer a higher tax on capital income, which implies a lower labor income tax. Since the optimal policy maximizes the average utility of newborn, their framework generates a higher optimal capital income tax. We confirm this by distributing accidental bequests equally to all population, in which case the optimal capital income tax increases to 42.4%.

Then why do we find, in our benchmark model with *rate of return heterogeneity*, the optimal capital income tax to be negative and the optimal wealth tax to be positive? In both Conesa et al. (2009) and in our model, a higher capital income tax reduces capital accumulation and leads to lower output. However, in our model, a higher capital income tax hurts productive agents disproportionately, leading to more misallocation, and further reductions in output.

Therefore, the capital income tax is much more distortionary in our environment with rate of return heterogeneity than in the environment in Conesa et al. (2009). With a wealth tax, the tax burden is shared between productive and unproductive agents, leading to a smaller misallocation and a lower decline in output as we increase wealth taxes. Thus, the government can increase the wealth tax without reducing output much, allowing it at the same time to reduce the labor income tax resulting in higher after-tax wages and thus higher welfare gains.

Transitions. Our analysis focuses on steady states and makes our results readily comparable to those in important recent papers on capital taxation such as Conesa et al. (2009). Steady-state welfare gains often are due to higher capital stocks, achieved through a transition period during which consumption is lower in order to allow the economy to invest towards building a larger capital stock. Thus, taking the transition period into account would usually lower the computed welfare gains of moving from one steady state to another. In our framework, with rate of return heterogeneity, however, the optimal wealth tax implies only a 2.76% increase in capital stock relative to the benchmark, and as a result it does not require much lower consumption during the transition. Most of the gain comes from a better allocation of capital. Thus we expect that not capturing the transition period would not change our results significantly. The case of an optimal wealth tax with a threshold—which delivers even larger welfare gains—requires an even smaller steady-state increase in capital stock of 0.41%. Therefore, although potentially interesting and worth exploring in future work, we conjecture that incorporating transitions is probably not going alter our main results.

In contrast, the analysis of the optimal capital income tax is likely to be substantially affected by the transition because it requires a 69% increase in the capital stock from the current US benchmark. This requires substantial saving and reduction in consumption during the transition, lowering the welfare gains.

Table 3.21: Robustness: **Optimal Tax** Experiments

	τ_k	τ_ℓ		τ_a	Top 1%	$\overline{CE}_2$ (%)	Vote (%)
				Baseline Model			
Baseline U.S.	25%	22.4%		–	0.36		
Opt. τ_k	–34.4%	36.0%		–	0.56	6.28	69.7
Opt. τ_a	–	14.1%		3.06%	0.47	9.61	60.7
				Progressive Labor Income Tax			
			ψ				
Benchmark	25%	15.0%	0.815		0.36		
Opt. τ_k	–38.8%	29.3%	0.720	–	0.61	9.31	52.1
Opt. τ_a	–	12.7%	0.720	2.40%	0.53	10.71	48.9
				Constant z over the life cycle			
Opt. τ_k	–2.33%	29.0%		–	0.47	3.27	83.1
Opt. τ_a	–	18.5%		2.21%	0.46	5.80	61.6
				No Financial Constraint			
Opt. τ_k	13.6%	26.0%		–	0.39	0.41	59.9
Opt. τ_a	–	22.7%		1.57%	0.42	1.43	56.6
				$\mu = 0.8$			
Opt. τ_k	-38.6%	37.7%		-	0.52	7.38	67.1
Opt. τ_a	-	18.6%		2.12%	0.44	8.32	66.0
				Estate Taxes			
Opt. τ_k	-32.2%	33.7%		—	0.56	9.26	72.5
Opt. τ_a	-	13.0%		3.12%	0.49	11.02	60.7
				Present Value			
Opt. τ_k	–18.3%	33.56%			0.46	4.16	70.3
Opt. τ_a	–	16.45%		2.64%	0.43	7.38	60.4
				Conesa, Kitao and Krueger(2009)			
Opt. τ_k	25.4%	22.33%		—	0.09	0.25	42.8%
Opt. τ_a	-	22.33%		1.93%	0.09	0.25	42.8%

3.8 Concluding Remarks

Many countries currently have or have had wealth taxes: France, Spain, Norway, Switzerland, Italy, Denmark, Germany, Finland, Sweden, among others. However, the rationale for such taxes are often vague—fairness, reducing inequality—and not studied formally. Here, we propose a case for wealth taxes based on efficiency and distributional benefits and quantitatively evaluate its impact. In particular, we analyze the quantitative implications of wealth taxation (tax on the stock of wealth) as opposed to capital income taxation (tax on the income flow from capital) in an overlapping-generations incomplete-markets model with rate of return heterogeneity across individuals. With such heterogeneity, capital income and wealth taxes have opposite implications for efficiency and some key distributional outcomes.

Under capital income taxation, entrepreneurs who are more productive, and therefore generate more income, pay higher taxes. Under wealth taxation, on the other hand, entrepreneurs who have similar wealth levels pay similar taxes regardless of their productivity, which expands the base and shifts the tax burden toward unproductive entrepreneurs. This reallocation increases aggregate productivity and output. In the simulated model calibrated to the U.S. data, a revenue-neutral tax reform that replaces capital income tax with a wealth tax raises welfare by about 8% in consumption-equivalent terms. Moving on to optimal taxation, the optimal wealth tax is positive, yields even larger welfare gains than the tax reform, and is preferable to optimal capital income taxes. Interestingly, optimal wealth taxes result in more even consumption and leisure distributions (despite the wealth distribution becoming more dispersed), which is the opposite of what optimal capital income taxes imply. Consequently, wealth taxes can yield both efficiency and distributional gains.

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Appendix A

Appendix to Chapter 1

A.1 Mathematical Preliminaries

I include definitions and theorems that are relevant for the proofs in the text and in Appendix A.2.

Definition 1. [**Probability Space**] A probability space is a triplet $(A, \mathcal{A}, \mu)$ of a set A, a σ-algebra $\mathcal{A}$ on that set and a probability measure $\mu : \mathcal{A} \to [0, 1]$. When the σ-algebra is understood (generally as the Borel σ-algebra) it is omitted.

Definition 2. [**Polish Space**] A set A is a polish space if it is separable (allows for a dense countable subset) and metrizable topological space (there exists at least one metric that induces the topology).

Definition 3. [**Coupling**] Let $(\mathcal{Y}, G)$ and $(\mathcal{X}, P)$ be two probability spaces. A coupling π of G and P is a joint distribution on $(\mathcal{X} \times \mathcal{Y})$ such that $\int_{\mathcal{X} \times Y} d\pi (x, y) = G (Y)$ for all $Y \in \mathcal{B} (\mathcal{Y})$ and $\int_{X \times \mathcal{Y}} d\pi (x, y) = P (X)$ for all $X \in \mathcal{B} (\mathcal{X})$, where $\mathcal{B} (A)$ denotes the Borel sets of A. So π gives G and P as marginals. Let $\Pi (P, G)$ be the set of all couplings of P and G. When the assignment is given by an assignment function the coupling is deterministic.

Definition 4. [**h-transform**] Let $h : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ be a function. The h-transform of a function $f : \mathcal{X} \to \mathbb{R}$ is given by:

$$f^h (y) = \sup_{x \in \mathcal{X}} \{h (x, y) - f (x)\}$$

Definition 5. [**h-convex**] A function $f : \mathcal{X} \to \mathbb{R}$ is said to be h-convex if there exists a function $g : \mathcal{Y} \to \mathbb{R}$ such that:

$$f(x) = \sup_{y \in \mathcal{Y}} \{h(x, y) - g(y)\}$$

Definition 6. [**h-subdifferential**] The h-subdifferential of a function $v : \mathcal{Y} \to \mathbb{R}$ is defined as the set $\partial^h v(y) = \left\{x \in \mathcal{X} \mid v(y) + v^h(x) = h(x, y)\right\}$.

The following theorem joins results from optimal transport on the existence of a solution to the Monge-Kantorovich problem and the applicability of Kantorovich's duality to the mass transportation problem:

Theorem 1. *Villani (2009, Thm. 5.10 and Thm 5.30) Let $(\mathcal{Y}, G)$ and $(\mathcal{X}, P)$ be two Polish probability spaces and let $h : \mathcal{X} \times \mathcal{Y} \to \mathbb{R} \cup \{-\infty\}$ be an upper semicontinuous function. Consider the optimal transport problem:*

$$\sup_{\pi \in \Pi(G,P)} \int_{\mathcal{X} \times \mathcal{Y}} h(x, y)\, d\pi(x, y)$$

where function $h(x, y)$ describes the gain (or surplus) of transporting a unit of mass from y to x, and $\Pi(G, P)$ denotes the set of couplings of G and P.

If there exist real valued lower semicontinuous functions $a \in L^1(P)$ and $b \in L^1(G)$:

$$\forall_{(x,y) \in \mathcal{X} \times \mathcal{Y}} \quad h(x, y) \leq a(x) + b(y)$$

then:

1. *There is duality:*

$$\begin{aligned}
\sup_{\pi \in \Pi(G,P)} \int_{\mathcal{X} \times \mathcal{Y}} h(x, y)\, d\pi(x, y) &= \inf_{(\lambda, v) \in L^1(P) \times L^1(G)} \int_{\mathcal{X}} \lambda(x)\, dP(x) + \int_{\mathcal{Y}} v(y)\, dG(y) \\
&= \inf_{w \in L^1(P)} \int_{\mathcal{X}} \lambda(x)\, dP(x) + \int_{\mathcal{Y}} \lambda^h(y)\, dG(y) \\
&= \inf_{v \in L^1(G)} \int_{\mathcal{X}} v^h(x)\, dP(x) + \int_{\mathcal{Y}} v(y)\, dG(y)
\end{aligned}$$

   where $\Pi(G, P)$ is the set of couplings of G and P, the inf in the first line is subject to $\lambda(x) + v(y) \geq h(x, y)$ and f^h denotes the h-transform of function f:

$$f^h(y) = \sup_{x \in \mathcal{X}} h(x, y) - f(x)$$

   The functions w and v are h-convex since they are the h-transform of one another.

2. *If, furthermore, h is real valued ($h : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$) and the solution to the Monge-Kantorovich problem is finite* $\left(\max_{\pi \in \Pi(G,P)} \int_{\mathcal{X}\times\mathcal{Y}} h(x,y)\, d\pi(x,y) < \infty\right)$ *then there is a measurable h-monotone set $\Gamma \subset \mathcal{X} \times \mathcal{Y}$*[1] *such that for any $\pi \in \Pi(G,P)$ the following statements are equivalent:*
   - (a) *π is optimal.*
   - (b) *π is h-cyclically monotone.*
   - (c) *There is a h-convex function λ such that $\lambda(x) + \lambda^h(y) = h(x,y)$ π-almost surely.*
   - (d) *There exist $\lambda : \mathcal{X} \to \mathbb{R}$ and $v : \mathcal{Y} \to \mathbb{R}$ such that $\lambda(x) + v(y) \geq h(x,y)$ with equality π-almost surely.*
   - (e) *π is concentrated in Γ.*
3. *If, h is real valued ($h : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$) and there are functions $c \in L^1(P)$ and $d \in L^1(G)$ such that:*

$$\forall_{(x,y)\in\mathcal{X}\times\mathcal{Y}} \quad c(x) + d(y) \leq h(x,y)$$

   then the dual problem has a solution. There is a function w that attains the infimum.
4. *(this part from Villani (2009, Thm. 5.30))If:*
   - (a) *h is real valued ($h : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$)*
   - (b) *the solution to the Monge-Kantorovich problem is finite:*

$$\max_{\pi\in\Pi(G,P)} \int_{\mathcal{X}\times\mathcal{Y}} h(x,y)\, d\pi(x,y) < \infty$$

   - (c) *for any h-convex function $v : \mathcal{Y} \to \mathbb{R} \cup \{-\infty\}$ the subdifferential $\partial^h v(y)$ is single valued G-almost everywhere*

   Then
   - (a) *there is a unique (in law) optimal coupling π of (G,P).*
   - (b) *the optimal coupling is deterministic: $T : \mathcal{Y} \to \mathcal{X}$.*
   - (c) *the optimal coupling is characterized by the existence of a function h- convex function v such that $T(y) = \partial^h v(y)$.*

Finally, Reynold's transport theorem is used extensively in the text:

Theorem 2. ***[Reynolds' Transport Theorem]*** *The rate of change of the integral of a scalar function f within a volume V is equal to the volume integral of the change of f, plus the boundary integral of the rate at which f flows though the boundary ∂V of outward unit*

[1] If a, b and h are continuous then Γ is closed.

normal n:

$$\nabla \int_V f(x)\, dV = \int_V \nabla f(x)\, dV + \int_{\partial V} f(x)\,(\nabla x \cdot n)\, dA$$

A.2 Proofs

Outline of the poof (Proposition 1) As is common in assignment problems, I first relax the problem in 1.4 to allow for non-deterministic assignments, see Kantorovich (2006) and Koopmans and Beckmann (1957). An assignment is then a joint measure over workers/task pairs: $\pi : \mathcal{X} \times \mathcal{B}(\mathcal{Y}) \to \mathbb{R}_+$, where $\mathcal{B}(\mathcal{Y})$ denotes the Borel sets of $\mathcal{Y}$. An assignment π is deemed feasible if it is a coupling of measures P and G, see definition 3 in Appendix A.1. In terms of the assignment problem π must guarantee that workers have enough time to perform all the time demanded by their occupations, and each task is completed at most once. Letting $\Pi(P, G)$ be the set of feasible assignments:

$$\pi \in \Pi(P, G) \longleftrightarrow \forall_n \int_{\mathcal{Y}} d\pi(x_n, y) \leq p_n \qquad \forall_{Y \in \mathcal{B}(\mathcal{Y})} \sum_{n=1}^{N} \int_{y \in Y} d\pi(x_n, y) \leq G(Y) \tag{A.1}$$

Note that the second condition can be simplified to: $\sum_{n=1}^{N} \pi(x_n, \{y\}) \leq g(y)$.
The problem is now to choose a coupling $\pi \in \Pi(P, G)$ to maximize output. I further simplify the problem by applying natural logarithm to the objective function. Doing so reveals the linearity of the problem in the choice variable π. The relaxed optimization problem is:

$$\max_{\pi \in \Pi(P,G)} \sum_{n=1}^{N} \int_{\mathcal{Y}} \ln q(x_n, y)\; d\pi(x_n, y) \tag{A.2}$$

Lemma 1 applies Theorem 5.10 of Villani (2009) to establish duality for the problem:

$$\max_{\pi \in \Pi} \sum_{n=1}^{N} \int_{\mathcal{Y}} \ln q(x_n, y)\; d\pi(x_n, y) = \inf_{(\lambda, v) \in \mathbb{R}^N \times L^1(G), w_n + v(y) \geq \ln q(x_n, y)} \sum_{n=1}^{N} \lambda_n p_n + \int_{\mathcal{Y}} \nu(y)\, dG \tag{A.3}$$

$$= \inf_{\lambda \in \mathbb{R}^N} \sum_{n=1}^{N} \lambda_n p_n + \int_{\mathcal{Y}} \max_n \{\ln q(x_n, y) - \lambda_n\}\, dG$$

λ and ν are the multipliers (or potentials) of the problem. Lemma 2 establishes that a solution to the dual problem $(\lambda^\star, v^\star)$ exists. The levels of $\lambda^\star$ and $\nu^\star$ are only determined up to an additive constant. Both the assignment and the value of the dual problem do not change if λ is increased by a constant κ for all workers and ν decreased by the same amount for all tasks. I normalize the value of the minimum $\lambda^\star$ to zero. This is convenient when relating the value of $\lambda^\star$ to the marginal product of workers and the wages in the decentralization of the optimal assignment.

The first two conditions on the production function q ensure that the value of the primal problem (A.2) and the dual problem (A.3) are finite, this is the key step in verifying the conditions for Theorem 5.10 of Villani (2009). In particular, the first condition avoids indeterminacies when evaluating the natural logarithm of q for any worker/task pair.

The solution to the dual problem provides a way to construct the optimal assignment $T^\star$. Lemma 3 applies Theorem 5.30 of Villani (2009) to construct $T^\star$ as the sub-differential of $v^\star$. The third condition on the production function q is crucial to establish single-valuedness of the sub-differential of $v^\star$. This gives the formula for the optimal assignment in (1.5). Galichon (2016, Ch. 5.3) presents an algorithm to solve the dual problem in (A.3).

I now turn to the general proof of the problem.

General setting Consider the set up of Section 1.2. There are N types of workers $\{x_1, \ldots x_N\} \equiv \mathcal{X}$, there is a mass p_n of workers of type x_n. The mass of workers is described by a (discrete) measure P so that $P(x_n) = p_n$. There is a continuum of tasks $y \in \mathcal{Y}$ distributed continuously according to an absolutely continuous measure $G : \mathcal{B}(\mathcal{Y}) \to \mathbb{R}_+$. $\mathcal{Y}$ is assumed compact.

Output is produced by completing tasks. A worker of type x_n performing task y produced $q(x_n, y)$. $q : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ is a real-valued function. Output for all worker/task pairs is aggregated into a final good:

$$F(\pi) = \begin{cases} \left(\sum_{n=1}^N \int (q(x_n, y))^{\frac{\sigma-1}{\sigma}} d\pi(x_n, y)\right)^{\frac{\sigma}{\sigma-1}} & \text{if } \sigma > 1 \\ \exp\left(\sum_{n=1}^N \int_{\mathcal{Y}} \ln q(x_n, y)\, d\pi(x_n, y)\right) & \text{if } \sigma = 1 \end{cases}$$

where $\pi \in \Pi(P, G)$ is a coupling of P and G (see definition 3). The coupling π describes the assignment: a mass $\pi(x, y)$ of workers of type x is assigned to task y.

The problem is to maximize output of the final good by choosing an assignment of tasks to workers π. I first transform the objective function so that the problem takes the form of a Monge-Kantorovich problem:

$$\max_{\pi \in \Pi(P,G)} \int h(x, y|\sigma)\, d\pi(x, y) \tag{A.4}$$

where $h(x,y|\sigma) = \begin{cases} (q(x,y))^{\frac{\sigma-1}{\sigma}} & \text{if } \sigma > 1 \\ \ln q(x,y) & \text{if } \sigma = 1 \end{cases}$.

The following proposition establishes duality for this problem:

Lemma 1. *If q satisfies the following properties:*

1. *$\sigma > 1$ or all workers can produce in some task: $\forall_x \exists y \quad q(x,y) > 0$*
2. *$q(x,\cdot)$ is upper-semicontinuous in y given $x \in \mathcal{X}$.*

Then, the following equalities hold:

$$\max_{\pi \in \Pi(P,G)} \int (h(x,y|\sigma))^{\frac{\sigma-1}{\sigma}} d\pi(x,y) = \inf_{(\lambda,v)\in\mathbb{R}^N \times L^1(G),} \sum_{n=1}^{N} \lambda_n p_n + \int_{\mathcal{Y}} v(y)\, dG(y)$$
$$= \inf_{\lambda \in \mathbb{R}^N} \sum_{n=1}^{N} \lambda_n p_n + \int_{\mathcal{Y}} \max_n \{q(x_n,y|\sigma) - \lambda_n\}\, dG(y)$$

where the inf in the first line is subject to $\lambda_n + v(y) \geq q(x_n,y|\sigma)$.

Proof. This follows from applying theorem 1 (Villani, 2009, Thm. 5.10). Note that $\mathcal{Y} \subset \mathbb{R}^n$ and $\mathcal{X}$ is finite they are both Polish spaces. $h(x,y|\sigma)$ is upper semicontinuous because $f(x) = x^{\frac{\sigma-1}{\sigma}}$ and $f(x) = \ln x$ are continuous and monotone increasing, and q is upper semicontinuous.
It is left to verify that there exist real valued lower semicontinuous functions $a \in L^1(P)$ and $b \in L^1(G)$:

$$\forall_{(x,y)\in\mathcal{X}\times\mathcal{Y}} \quad h(x,y|\sigma) \leq a(x) + b(y)$$

For this let $a(x) = \max_{y\in\mathcal{Y}} \{h(x,y|\sigma)\}$ and $b(y) = 0$. The max in the definition of a is well defined because h is upper semicontinuous and $\mathcal{Y}$ is compact, furthermore a is finite (either $\sigma > 1$ or, if $\sigma = 1$, h is finite for at least some y guaranteeing a a final value). Function a is immediately continuous with respect to the discrete topology. The desired equalities follow from part 1 of Theorem 1. □

The dual problem is then to find a value associated with each type of worker $\{\lambda_1, \ldots, \lambda_N\}$. The problem is:

$$\inf_{\lambda\in\mathbb{R}^N} \sum_{n=1}^{N} \lambda_n p_n + \int_{\mathcal{Y}} v(y)\, dG(y) \qquad \text{where: } v(y) = \max_{n\in\{1,\ldots,N\}} \{h(x,y|\sigma) - \lambda_n\} \tag{A.5}$$

I show that the dual problem has a solution and I use that solution to construct a solution to the Monge-Kantorovich problem in (A.4). Furthermore, the solution will take the form of a deterministic transport, and the implied assignment function is the solution to the problem (1.4) in the main text. Part 3 of Theorem 1 establishes that solution to the dual problem (A.5) exists.

Lemma 2. *If q satisfies the following properties:*

1. *$\sigma > 1$ or all workers can produce in all tasks: $q(x,y) > 0$ for all pairs $(x,y) \in \mathcal{X} \times \mathcal{Y}$.*
2. *$q(x,\cdot)$ is upper-semicontinuous in y given $x \in \mathcal{X}$.*

Then there exists $\lambda^\star \in \mathbb{R}^N$ such that:

$$\lambda^\star \in \textit{argmin}_{\lambda \in \mathbb{R}^N} \sum_{n=1}^{N} \lambda_n p_n + \int_{\mathcal{Y}} \left(\max_{n \in \{1,\ldots,N\}} \{h(x,y|\sigma) - \lambda_n\} \right) dG(y)$$

Proof. This follows from applying part 3 of theorem 1 (Villani, 2009, Thm. 5.10). The function $h(x,y|\sigma)$ is required to be real valued. When $\sigma > 1$ this is verified since q is real valued. When $\sigma = 1$ it is verified under the additional condition that $q(x,y) > 0$ for all $(x,y) \in \mathcal{X} \times \mathcal{Y}$.
It is left to find functions $c \in L^1(P)$ and $d \in L^1(G)$ such that:

$$\forall_{(x,y) \in \mathcal{X} \times \mathcal{Y}} \quad c(x) + d(y) \le h(x,y|\sigma)$$

For this let $c(x) = 0$ and $d(y) = \min_n \{h(x,y|\sigma)\}$. The minimum is well defined since $\mathcal{X}$ is finite.

□

The final part of Proposition 1 is obtained from applying Theorem 5.30 of Villani (2009), reproduced as part 4 of Theorem 1. The result is established under the conditions that both $(F(x,y))^{\frac{\sigma-1}{\sigma}}$ and the Monge-Kantorovich problem (A.4) have finite value and the F-subdifferential of w is single-valued G-almost everywhere.

Lemma 3. *If q is such that:*

1. *$\sigma > 1$ or all workers can produce in all tasks: $q(x,y) > 0$ for all pairs $(x,y) \in \mathcal{X} \times \mathcal{Y}$.*
2. *$q(x,\cdot)$ is upper-semicontinuous in y given $x \in \mathcal{X}$.*
3. *q discriminates across workers in almost all tasks: if $q(x_n,y) = q(x_m,y)$ then $x_n = x_m$ G-a.e.*

Then there exists $\lambda^\star \in \mathbb{R}^N$ that solves the dual problem (A.5). Moreover, let T be defined as:

$$T(y) = \textit{argmax}_{x \in \mathcal{X}} \left\{ h(x,y|\sigma) - \lambda^\star_{n(x)} \right\}$$

T is single-valued G-almost everywhere and it induces a deterministic coupling

$$\pi^\star : \mathcal{X} \times \mathcal{B}(\mathcal{Y}) \to \mathbb{R}_+$$

that is the unique (in law) solution to the Monge-Kantorovich problem (A.4). $\pi^\star$ is:

$$\pi^\star(x_n, Y) = \int_{Y \cap T^{-1}(x_n)} dG$$

Function T is an assignment function and it is the solution to the Monge transportation problem (1.4).

Proof. The proof follows from applying part 4 of Theorem 1 (from Villani (2009, Thm. 5.30)). Finiteness of $h(x, y|\sigma)$ is guaranteed if $\sigma > 1$, or if $\sigma = 1$ and $q(x, y) > 0$ for all pairs $(x, y) \in \mathcal{X} \times \mathcal{Y}$. Finiteness of the value of the Monge-Kantorovich problem is guaranteed since $\mathcal{Y}$ and $\mathcal{X}$ are both compact, and q is upper semicontinuous on y.
It is left to verify that for any h-convex function $v : \mathcal{Y} \to \mathbb{R} \cup \{-\infty\}$ the h-subdifferential $\partial^h v(y)$ is single valued G-almost everywhere. The h- subdifferential for a given y is given by:

$$\partial^h v(y) = \left\{x \in \mathcal{X} \mid v^h(x) + v(y) = h(x, y|\sigma)\right\} \qquad \text{where} \qquad v^h(x) = \sup_y \{h(x, y|\sigma) - v(y)\}$$

Since v is h-convex we can instead use its conjugate function $v^h(x_n) = \lambda_n$. Then the h-subdifferential is then equivalently given by:

$$\partial^h v(y) = \text{argmax}_{x \in \mathcal{X}} \left\{h(x, y|\sigma) - \lambda_{n(x)}\right\}$$

Since $q(\cdot, y)$ is injective in x given y G-a.e., and $\mathcal{X}$ is finite, we get that $\partial^h v(y)$ is generically a singleton.

□

The following lemma establishes the relation between the multipliers of the transformed problem (A.4) and multipliers of the original problem (A.2).

Lemma 4. *Consider two constrained maximization problems:*

$$V(m) = \max_x \quad F(x) \qquad s.t.\ h(x) = m \tag{A.6}$$

$$W(m) = \max_x \quad g(F(x)) \qquad s.t.\ h(x) = m \tag{A.7}$$

where $F : \mathcal{X} \to \mathbb{R}$, $h : \mathcal{X} \to \mathbb{R}^n$, $m \in \mathbb{R}^n$ and $g : \mathbb{R} \to \mathbb{R}$ is strictly monotone. Let $\lambda \in \mathbb{R}^n$ be the multiplier associated with the constraints in (A.6), and $\mu \in \mathbb{R}^n$ the multiplier associated with the constraints in (A.7). Then: $\mu = g^{'}(F(x^\star))\lambda$, where $x^\star$ is a solution for (A.6) and (A.7).

Proof. Because g is strictly monotone both problems have the same argmax, call it $x^\star(m)$. The value of each problem is:

$$V(m) = F(x^\star(m)) \qquad W(m) = g(F(x^\star(m)))$$

By the envelope theorem (Milgrom and Segal, 2002) we know that:

$$\lambda = \frac{\partial V(m)}{\partial m} = \frac{\partial F(x^\star)}{\partial x}\frac{\partial x^\star(m)}{\partial m} \qquad \mu = \frac{\partial W(m)}{\partial m} = \frac{\partial g(F(x^\star))}{\partial F}\frac{\partial F(x^\star)}{\partial x}\frac{\partial x^\star(m)}{\partial m}$$

Joining gives the result: $\mu = g^{'}(F(x^\star))\lambda$.

$\square$

I now present the proof for the differentiability of demand:

Proposition 2. *Let $\lambda \in \mathbb{R}^N$ be a vector of multipliers. If q is continuous then D_n is continuously differentiable with respect to w and:*

i $\frac{\partial D_n}{\partial w_m} = \frac{area(\mathcal{Y}_n(w) \cap \mathcal{Y}_m(w))}{2\sqrt{(x_n - x_m)' A' A (x_n - x_m)}} \geq 0$

ii $\frac{\partial D_n}{\partial w_n} = -\sum_{m \neq n} \frac{\partial D_m}{\partial w_n} < 0$

Proof. First note that since the space of tasks $\mathcal{Y}$ is fixed it holds that $\sum_{n=1}^{N} D_n = \int_{\mathcal{Y}} dy$ so the sum of demands is constant. Then:

$$\frac{\partial D_n}{\partial w_n} + \sum_{m \neq n} \frac{\partial D_m}{\partial w_n} = 0$$

which gives part (ii) of the proposition, the relation between the demand's own price derivative and the cross derivatives of other demands.

The rest of the proof follows from an application of Reynolds' Transport Theorem (Theorem 2). In order to apply Reynolds' theorem recall that $D_m = \int_{\mathcal{Y}_m} \rho(y)\, dy$, where ρ is the density of tasks in the space. In our case $\rho(y) = 1$. So the volume is $\mathcal{Y}_m$ and the function is the density of tasks.

The second term in the theorem measures the rate at which the density flows in and out of the volume. The density flows out and into other workers as tasks are reassigned. Consider the flow into of $\mathcal{Y}_m$ and out of $\mathcal{Y}_k$. The flow is in the direction $\frac{A(x_k - x_m)}{\sqrt{(x_k - x_m)' A' A (x_k - x_m)}}$ and through the shared boundary of $\mathcal{Y}_m$ and $\mathcal{Y}_k$, given by $\mathcal{Y}_m \cap \mathcal{Y}_k$. Note that when prices change the hyperplanes that define the boundaries of the demand sets move in parallel.

Applying the theorem:

$$\frac{\partial D_m}{\partial w_n} = \int_{\mathcal{Y}_m} \frac{\partial \rho(y)}{\partial w_n} dy + \sum_{k \neq m} \int_{\mathcal{Y}_m \cap \mathcal{Y}_k} \rho(y) \left(\frac{\partial y \cdot \frac{A(x_k - x_m)}{\sqrt{(x_k - x_m)' A' A (x_k - x_m)}}}{\partial w_n} \right) dy$$

Note that for all $y \in \mathcal{Y}_m \cap \mathcal{Y}_k$ lie in a plane perpendicular to $A(x_k - x_m)$. Then they can be always expressed as $y = y_\lambda + a\vec{v}$ where $a \in \mathbb{R}$, $\vec{v}$ is a vector perpendicular to $A(x_k - x_m)$ and $y_\lambda = (1 - \lambda) x_k + \lambda x_m$ is such that $y_\lambda \in \mathcal{Y}_m \cap \mathcal{Y}_k$. Then the change $y \in \mathcal{Y}_m \cap \mathcal{Y}_k$ is equal to the change in y_λ.

$$\frac{\partial D_m}{\partial w_n} = \sum_{k \neq m} \int_{\mathcal{Y}_m \cap \mathcal{Y}_k} \rho(y) \left(\frac{\partial y_\lambda \cdot \frac{A(x_k - x_m)}{\sqrt{(x_k - x_m)' A' A (x_k - x_m)}}}{\partial w_n} \right) dy$$

The value of λ is obtained from the equation for the hyperplane that defines $\mathcal{Y}_m \cap \mathcal{Y}_k$:

$$\lambda = \frac{(x_m - x_k)' A (x_m - x_k) + w_m - w_k}{2 (x_m - x_k)' A (x_m - x_k)}$$

so:

$$\frac{\partial y_\lambda \cdot \frac{x_k - x_m}{\sqrt{(x_k - x_m)'(x_k - x_m)}}}{\partial w_n} = \begin{cases} \frac{1}{2\sqrt{(x_n - x_m)' A' A (x_n - x_m)}} & \text{if } k = n \\ 0 & \text{otw} \end{cases}$$

Replacing:

$$\frac{\partial D_m}{\partial w_n} = \frac{\int_{\mathcal{Y}_m \cap \mathcal{Y}_n} \rho(y)\, dy}{2\sqrt{(x_n - x_m)'(x_n - x_m)}} = \frac{\text{area}(\mathcal{Y}_n(w) \cap \mathcal{Y}_m(w))}{2\sqrt{(x_n - x_m)'(x_n - x_m)}}$$

which completes the proof.

$\square$

Proposition 3. *Consider the automation problem in 2.1 and let $\mu \in \mathbb{R}^{N+1}$ characterize an assignment according to 2.3. If q is differentiable then the first order conditions of the problem are:*

$$F_R(\mu, r) \int_{\mathcal{Y}_R} \frac{\partial \ln q(r, y)}{\partial r} dy - \frac{\partial \Omega(r, p_r)}{\partial r} = 0 \qquad [r]$$

$$F_R(\mu, r)\, \mu_R - \frac{\partial \Omega(r, p_r)}{\partial p_r} = 0 \qquad [p_r]$$

Proof. After replacing T_R for μ in the problem, and abusing notation, the corresponding Lagrangian is:

$$\max_{\{r, p_r, \mu, \Lambda\}} \mathcal{L} = F_R(\mu, r) - \Omega(r, p_r) + \sum_{n=1}^{N} \Lambda_n (p_n - D_n) + \Lambda_R (p_r - D_R) \tag{A.8}$$

The multipliers of the workers/robot capacity constraints are given by the vector $\Lambda \in \mathbb{R}^{N+1}$. The first order condition of interest is with respect to the skills of the robot:

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{\partial F_R(\mu, r)}{\partial r} - \frac{\partial \Omega(r, p_r)}{\partial r} - \sum_{n=1}^{N} \Lambda_n \frac{\partial D_n}{\partial r} - \Lambda_R \frac{\partial D_R}{\partial r} \tag{A.9}$$

Following Goes et al. (2012) and using the result in Lemma 4 the first order condition becomes:

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{\partial F_R(\mu, r)}{\partial r} - \frac{\partial \Omega(r, p_r)}{\partial r} - F_R(\mu, r) \left(\sum_{n=1}^{N} \mu_n \frac{\partial D_n}{\partial r} - \mu_R \frac{\partial D_R}{\partial r} \right) \tag{A.10}$$

I proceed by computing separately the first term of the first order condition:

$$\frac{\partial F_R(\mu, r)}{\partial r} = F_R(\mu, r) \left(\sum_{n=1}^{N} \frac{\partial \int_{\mathcal{Y}_n} \ln q(x_n, y)\, dy}{\partial r} + \frac{\partial \int_{\mathcal{Y}_R} \ln q(r, y)\, dy}{\partial r} \right)$$

Each of the derivatives follows from Reynold's theorem.

$$\begin{aligned} \frac{\partial \int_{\mathcal{Y}_n} \ln q(x_n, y)\, dy}{\partial r} &= \int_{\mathcal{Y}_n} \frac{\partial \ln q(x_n, y)}{\partial r} dy + \int_{\mathcal{Y}_n \cap \mathcal{Y}_R} \ln q(x_n, y) \frac{\partial y \cdot c_{nr}}{\partial r} dy \\ &= \int_{\mathcal{Y}_n \cap \mathcal{Y}_R} \ln q(x_n, y) \frac{\partial y \cdot c_{nr}}{\partial r} dy \end{aligned}$$

where $c_{nr} = \frac{2A(x_n - r)}{\sqrt{(x_n - r)' A (x_n - r)'}}$ is the normal vector to the direction in which the boundary is moving.

In a similar way:

$$\frac{\partial \int_{\mathcal{Y}_R} \ln q(r,y)\, dy}{\partial r} = \int_{\mathcal{Y}_R} \frac{\partial \ln q(r,y)}{\partial r} dy + \sum_{n=1}^{N} \int_{\mathcal{Y}_n \cap \mathcal{Y}_R} \ln q(x_n, y) \frac{\partial y \cdot c_{rn}}{\partial r} dy$$

where $c_{rn} = -c_{nr}$. Joining and reorganizing we get:

$$\frac{1}{F_R(\mu,r)} \frac{\partial F_R(\mu,r)}{\partial r} = \int_{\mathcal{Y}_R} \frac{\partial \ln q(r,y)}{\partial r} dy + \sum_{n=1}^{N} \int_{\mathcal{Y}_n \cap \mathcal{Y}_R} (\ln q(x_n,y) - \ln q(r,y)) \frac{\partial y \cdot c_{nr}}{\partial r} dy$$

Note now that by the definition of the boundary $\ln q(x_n,y) - \ln q(r,y) = \mu_n - \mu_r$ for all $y \in \mathcal{Y}_n \cap \mathcal{Y}_R$. Then:

$$\frac{1}{F_R(\mu,r)} \frac{\partial F_R(\mu,r)}{\partial r} = \int_{\mathcal{Y}_R} \frac{\partial \ln q(r,y)}{\partial r} dy + \sum_{n=1}^{N} (\mu_n - \mu_r) \int_{\mathcal{Y}_n \cap \mathcal{Y}_R} \frac{\partial y \cdot c_{nr}}{\partial r} dy$$

Finally note that $\frac{\partial D_n}{\partial r} = \int_{\mathcal{Y}_n \cap \mathcal{Y}_R} \frac{\partial y \cdot c_{nr}}{\partial r} dy$, which follows from applying Reynold's Theorem (again) to D_n.

$$\frac{1}{F_R(\mu,r)} \frac{\partial F_R(\mu,r)}{\partial r} = \int_{\mathcal{Y}_R} \frac{\partial \ln q(r,y)}{\partial r} dy + \sum_{n=1}^{N} (\mu_n - \mu_r) \frac{\partial D_n}{\partial r}$$

When the location of the robot (r) is changed, there is a change in output due to the change in mismatch inside the region previously assigned to the robot ($\mathcal{Y}_R$), that is given by the first term. There is also a change in the demand for workers, only workers who are neighbors of the robot are affected. When their demand is affected, the demand of the robot changes in the opposite direction. The demand for worker n changes by $\frac{\partial D_n}{\partial r}$, that is valued by the planner at $\lambda_n - \lambda_r$. Recall that λ_n is the shadow price of the supply of a worker.
It is left to spell out the first term:

$$\begin{aligned} \int_{\mathcal{Y}_R} \frac{\partial \ln q(r,y)}{\partial r} dy &= \int_{\mathcal{Y}_R} \frac{\partial a_x^{'} r - (r-y)^{'} A (r-y)}{\partial r} dy \\ &= \int_{\mathcal{Y}_R} (a_x - 2Ar + 2Ay)\, dy \\ &= 2D_R \left(\frac{a_x}{2} - A(r - b_R) \right) \end{aligned}$$

where $b_R = \frac{\int_{\mathcal{Y}_R} y dy}{D_R}$ is the barycenter (centroid, average or center of mass) of the tasks assigned to r.
It is now possible to obtain the first order condition of the problem with respect to the location of the robot. Note that since the total demand is constant it holds that:

$$\frac{\partial D_R}{\partial r} = -\sum_{n=1}^{N} \frac{\partial D_n}{\partial r}$$

then:

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{\partial F_R(\mu,r)}{\partial r} - \frac{\partial \Omega(r,p_r)}{\partial r} - F_R(\mu,r) \left(\sum_{n=1}^{N} (\mu_n - \mu_R) \frac{\partial D_n}{\partial r} \right) \tag{A.11}$$

Replacing for $\frac{\partial F_R(\mu,r)}{\partial r}$ we get:

$$\frac{\partial \mathcal{L}}{\partial r} = 2F_R(\mu,r) D_R \left(\frac{a_x}{2} - A(r - b_R) \right) - \frac{\partial \Omega(r,p_r)}{\partial r} \tag{A.12}$$

The first order condition does not include the effect of r on the demand for workers since the gains cancel with the reductions/increases of slack in the feasibility constraints.
This is a necessary condition for an optimum. It does not fully characterize the solution. In fact, there can be, in general, multiple solutions to the problem. The first order condition is also silent about the location of the region assigned to r. Instead, it prescribes the relationship between the region's centroid and the location of r. It is convenient to see what happens when $a_x = 0$ and $\frac{\partial \Omega(r,p_r)}{\partial r} = 0$. Then the necessary condition reduces to make r equal to the barycenter of its region.
The first order condition with respect to p_r is:

$$\frac{\partial F}{\partial p_r} = F_R(\mu, r)\, \mu_R - \frac{\partial \Omega(r, p_r)}{\partial p_r}$$

The first order condition with respect to μ requires more work, but it follows from applying again Reynolds' Transport Theorem.

$$\frac{\partial F}{\partial \mu_n} = p_n - D_n$$

□

A.3 Marginal Product

The marginal product of a worker gives the change in output if more workers of that type are used in production. The change in output depends on the tasks that are assigned to additional workers.[2] Because of this, it is possible to define the marginal product at a given task, and under some initial assignment. In the main text, I consider the notion of equilibrium marginal products, where the assignment is not taken as given, but it is allowed to react optimally to changes in the supply of workers.

Consider the marginal product of a worker of type x_k at task $\overline{y}$, given an assignment T. Since task $\overline{y}$ has no mass, output does not change if the task is re-assigned to a worker of type x_k. The marginal product is measured by adding a mass of workers of type x_k and assigning them to a region around task $\overline{y}$, replacing the workers previously assigned to those tasks. The marginal product at $\overline{y}$ is obtained as the change in output when the mass of added workers tends to zero.

Proposition 4. ***[Marginal Product]*** *Let T be a deterministic assignment and fix a task $\overline{y} \in \mathcal{Y}_n^{\text{o}}$. The marginal product of a worker of type x_k at task $\overline{y}$ is:*

$$MP\left(x_k, \overline{y}|T\right) = F\left(T\right)\left(\ln q\left(x_k, \overline{y}\right) - \ln q\left(x_n, \overline{y}\right)\right)$$

where $F\left(T\right) = \exp\left(\int \ln q\left(T\left(y\right), y\right) dG\right)$ and $T\left(\overline{y}\right) = x_n$.

When task $\overline{y}$ is re-assigned from x_n to x_k output changes by $\ln q\left(x_k, \overline{y}\right) - \ln q\left(x_n, \overline{y}\right)$. The marginal product takes into account the opportunity cost of assigning task $\overline{y}$ to x_k, which comes from the capacity constraint of tasks. The derivative of output takes into account the scale of production at the current assignment. Task $\overline{y}$ is required to be in the interior of $\mathcal{Y}_n$ for technical reasons. If $\overline{y} \in \mathcal{Y}_n \cap \mathcal{Y}_m$ it becomes necessary to specify the region around $\overline{y}$ to which x_k will be assigned.

The proof of the result is complicated because the task $\overline{y}$ has dimension zero in the space of tasks, which has dimension $d \geq 1$. Before showing the general proof for the result, I consider the one-dimensional case where the argument is simpler. I further assume that $y \sim U\left(\left[0, 1\right]\right)$. When $d = 1$ the production function can be written as:

$$F\left(T\right) = \exp\left(\int_0^1 \ln q\left(T\left(y\right), y\right) dy\right)$$

Fix a task $\overline{y} \in \left(0, 1\right)$ and consider adding a mass ϵ of workers of type x_k. Workers are

[2] Unlike traditional production functions, the amount of an input used by the firm in production and what that input is used for are not the same.

assigned to the set $C_{\overline{y},\epsilon} = \left\{ y \,|\, |y - \overline{y}| < \frac{\epsilon}{2} \right\} = \left[\overline{y} - \frac{\epsilon}{2}, \overline{y} + \frac{\epsilon}{2} \right]$. The new assignment is:

$$T_{\epsilon}(y) = \begin{cases} T(y) & \text{if } y \notin C_{\overline{y},\epsilon} \\ 0 & \text{if } y \in C_{\overline{y},\epsilon} \wedge x \neq x_k \\ 1 & \text{if } y \in C_{\overline{y},\epsilon} \wedge x = x_k \end{cases}$$

The change in output is:

$$F(T_{\epsilon}) - F(T) = F(T) \left(\exp \left(\int_{\overline{y}-\frac{\epsilon}{2}}^{\overline{y}+\frac{\epsilon}{2}} (\ln q(x_k, y) - \ln q(T(y), y))\, dy \right) - 1 \right)$$

The marginal product is:

$$\text{MP}(x_k, \overline{y}|T) = \left. \frac{\partial F(R_{\epsilon})}{\partial \epsilon} \right|_{\epsilon=0} = \lim_{\epsilon \to 0} \frac{F(R_{\epsilon}) - F(T)}{\epsilon}$$

replacing and applying L'Hôpital's rule:

$$\text{MP}(x_k, \overline{y}|T) = F(R) \left. \frac{\partial \exp \left(\int_{\overline{y}-\frac{\epsilon}{2}}^{\overline{y}+\frac{\epsilon}{2}} (\ln q(x_k, y) - \ln q(T(y), y))\, dy \right)}{\partial \epsilon} \right|_{\epsilon=0}$$

The derivative follows from Leibniz's rule. Generically $\overline{y} \in \mathcal{Y}_n^{\text{o}}$ and:

$$\begin{aligned} \text{MP}(x_k, \overline{y}|T) = F(T) \Bigg[& \frac{1}{2} \left(\ln q \left(x_k, \overline{y} + \frac{\epsilon}{2} \right) - \ln q \left(T(y), \overline{y} + \frac{\epsilon}{2} \right) \right) \\ & + \frac{1}{2} \left(\ln q \left(x_k, \overline{y} - \frac{\epsilon}{2} \right) - \ln q \left(T(y), \overline{y} - \frac{\epsilon}{2} \right) \right) \Bigg]_{\epsilon=0} \\ = F(T) & (\ln q(x_k, \overline{y}) - \ln q(x_n, \overline{y})) \end{aligned}$$

If $\overline{y} \in \mathcal{Y}_n \cap \mathcal{Y}_m$ the marginal product takes into account that x_k replaces different types of workers around $\overline{y}$:

$$\text{MP}(x_k, \overline{y}|T) = F(T) \left(\ln q(x_k, \overline{y}) - \frac{\ln q(x_n, \overline{y}) + \ln q(x_m, \overline{y})}{2} \right)$$

In multiple dimensions, the treatment of the boundary cases becomes intractable, except in very specific cases for which similar expressions are obtained.
I now provide the general proof of the result.

Proof. Recall that the space of skills is of dimension d. Changing the assignment of tasks to workers in any region of dimension less than d will have no impact on output. To compute the effect on output of the added workers it is necessary to proceed one dimension at a time. Consider a region formed as a hypercube around $\overline{y}$, with sides of length ϵ_i, denote this region by $C_{\overline{y},\epsilon} = \left\{ y \,|\, \forall_i \left| y_i - \overline{y}_i \right| \leq \frac{\epsilon_i}{2} \right\}$. Note that as all $\epsilon_i \to 0$ the region $C_{\overline{y},\epsilon} \to \{\overline{y}\}$. The assignment is modified as in the one-dimensional example:

$$T_\epsilon(y) = \begin{cases} T(y) & \text{if } y \notin C_{\overline{y},\epsilon} \\ 0 & \text{if } y \in C_{\overline{y},\epsilon} \wedge x \neq x_k \\ 1 & \text{if } y \in C_{\overline{y},\epsilon} \wedge x = x_k \end{cases}$$

The difference in production between the two assignments is:

$$F(T_\epsilon) - F(T) = F(T) \left(\exp \left(\int_{\overline{y}_1 - \frac{\epsilon_1}{2}}^{\overline{y}_1 + \frac{\epsilon_1}{2}} \cdots \int_{\overline{y}_d - \frac{\epsilon_d}{2}}^{\overline{y}_d + \frac{\epsilon_d}{2}} \left(\ln q(x_k, y) - \ln q(T(y), y) \right) dy \right) - 1 \right)$$

I proceed by computing the change in output when the region $C_{\overline{y},\epsilon}$ changes. The change has to be computed one dimension at a time. If all dimensions are changed simultaneously the change in F goes to zero (this can be verified directly using Reynold's transport theorem- Theorem 2). The change in output when $C_{\overline{y},\epsilon}$ changes in the d^{th} dimension is:

$$\begin{aligned} \frac{\partial F(T_\epsilon)}{\partial \epsilon_d} = F(T) \Bigg(\int_{\overline{y}_1 - \frac{\epsilon_1}{2}}^{\overline{y}_1 + \frac{\epsilon_1}{2}} \cdots \int_{\overline{y}_{d-1} - \frac{\epsilon_d}{2}}^{\overline{y}_{d-1} + \frac{\epsilon_d}{2}} \\ \frac{1}{2} \left(\ln q \left(x_k, \begin{pmatrix} y_1 \\ \vdots \\ \overline{y}_d + \frac{\epsilon}{2} \end{pmatrix} \right) - \ln q \left(T \begin{pmatrix} y_1 \\ \vdots \\ \overline{y}_d + \frac{\epsilon}{2} \end{pmatrix}, \begin{pmatrix} y_1 \\ \vdots \\ \overline{y}_d + \frac{\epsilon}{2} \end{pmatrix} \right) \right) \\ + \frac{1}{2} \left(\ln q \left(x_k, \begin{pmatrix} y_1 \\ \vdots \\ \overline{y}_d - \frac{\epsilon}{2} \end{pmatrix} \right) - \ln q \left(T \begin{pmatrix} y_1 \\ \vdots \\ \overline{y}_d - \frac{\epsilon}{2} \end{pmatrix}, \begin{pmatrix} y_1 \\ \vdots \\ \overline{y}_d - \frac{\epsilon}{2} \end{pmatrix} \right) \right) dy_1 \ldots dy_{d-1} \Bigg) \end{aligned}$$

Applying the same procedure iteratively we obtain the change in output as x_k is assigned to tasks around $\overline{y}$ in all directions:

$$\mathrm{MP}(x_k, \overline{y} | T) = \left. \frac{\partial^d F(T_\epsilon)}{\partial \epsilon_1 \cdots \partial \epsilon_d} \right|_{\epsilon = 0} = F(T) \left(\ln q(x_k, \overline{y}) - \ln q(x_n, \overline{y}) \right)$$

□

Appendix B

Appendix to Chapter 2

B.1 List of Cognitive and Manual Attributes

Cognitive		Manual	
Code	Attribute	Code	Attribute
Worker Characteristics - Abilities			
1A1a2	Written Comprehension	1A1e1	Speed of Closure
1A1a4	Written Expression	1A1e2	Flexibility of Closure
1A1b1	Fluency of Ideas	1A1e3	Perceptual Speed
1A1b2	Originality	1A1f1	Spatial Orientation
1A1b3	Problem Sensitivity	1A1f2	Visualization
1A1b4	Deductive Reasoning	1A1g1	Selective Attention
1A1b5	Inductive Reasoning	1A1g2	Time Sharing
1A1b6	Information Ordering	1A1g1	Selective Attention
1A1b7	Category Flexibility	1A1g2	Time Sharing
1A1c1	Mathematical Reasoning	1A2a1	Arm-Hand Steadiness
1A1c2	Number Facility	1A2a2	Manual Dexterity
1A1d1	Memorization	1A2a3	Finger Dexterity
		1A2b1	Control Precision
		1A2b2	Multi-limb Coordination
		1A2b3	Response Orientation
		1A2b4	Rate Control
		1A2c1	Reaction Time
		1A2c2	Wrist-Finger Speed

		1A2c3	Speed of Limb Movement
		1A3a	Physical Strength Abilities
		1A3a1	Static Strength
		1A3a2	Explosive Strength
		1A3a3	Dynamic Strength
		1A3a4	Trunk Strength
		1A3b1	Stamina
		1A3c1	Extent Flexibility
		1A3c2	Dynamic Flexibility
		1A3c3	Gross Body Coordination
		1A3c4	Gross Body Equilibrium
Worker Characteristics - Interests			
1B1b	Investigative		
1B1c	Artistic		
1B1e	Enterprising		
Worker Requirements - Basic Abilities			
2A1a	Reading Comprehension		
2A1b	Active Listening		
2A1c	Writing		
2A1d	Speaking		
2A1e	Mathematics		
2A1f	Science		
2A2a	Critical Thinking		
2A2b	Active Learning		
2A2c	Learning Strategies		
2A2d	Monitoring		
Worker Requirements - Cross-Functional Skills			
2B2i	Complex Problem Solving	2B3d	Installation
2B3a	Operations Analysis	2B3h	Operation and Control
2B3b	Technology Design	2B3j	Equipment Maintenance
2B3c	Equipment Selection	2B3l	Repairing
2B3e	Programming		
2B3g	Operation Monitoring		
2B3k	Troubleshooting		
2B4e	Judgment and Decision Making		

2B4g	Systems Analysis		
2B4h	Systems Evaluation		
	Resource Management Skills		
2B5a	Time Management		
2B5b	Management of Financial Resources		
2B5c	Management of Material Resources		
2B5d	Management of Personnel Resources		
	Knowledge		
2C1a	Administration and Management	2C1b	Clerical
2C1c	Economics and Accounting	2C2a	Production and Processing
2C3a	Computers and Electronics	2C3c	Design
2C3b	Engineering and Technology	2C3d	Building and Construction
2C4a	Mathematics	2C3e	Mechanical
2C4b	Physics	2C9a	Telecommunications
2C4c	Chemistry	2C10	Transportation
2C4d	Biology		
2C4f	Sociology and Anthropology		
2C4g	Geography		
2C5a	Medicine and Dentistry		
2C6	Education and Training		
2C7a	English Language		
2C7b	Foreign Language		
2C7c	Fine Arts		
2C7d	History and Archeology		
2C7e	Philosophy and Theology		
2C8a	Public Safety and Security		
2C8b	Law and Government		
	Generalized Work Activities		
4A1b2	Inspecting Equipment- Structures- or Material	4A1a2	Monitor Processes- Materials- or Surroundings
4A1b3	Estimating the Quantifiable Characteristics of Products- Events- or Information	4A1b1	Identifying Objects- Actions- and Events
4A2a1	Judging the Qualities of Things- Services- or People	4A3a1	Performing General Physical Activities

4A2a2	Processing Information	4A3a2	Handling and Moving Objects
4A2a3	Evaluating Information to Determine Compliance with Standards	4A3a3	Controlling Machines and Processes
4A2a4	Analyzing Data or Information	4A3a4	Operating Vehicles- Mechanized Devices- or Equipment
4A2b1	Making Decisions and Solving Problems	4A3b4	Repairing and Maintaining Mechanical Equipment
4A2b2	Thinking Creatively	4A3b5	Repairing and Maintaining Electronic Equipment
4A2b3	Updating and Using Relevant Knowledge	4A3b6	Documenting/Recording Information
4A2b4	Developing Objectives and Strategies	4A4c1	Performing Administrative Activities
4A2b5	Scheduling Work and Activities		
4A2b6	Organizing- Planning- and Prioritizing Work		
4A3b1	Interacting With Computers		
4A4c3	Monitoring and Controlling Resources		

B.2 List of Occupational Groups

The following table shows the classification of 2010 SOC occupations into the five groups used in Section 2.3. The two tables below further divide SOC occupations 11 and 19 into the groups used in x_4 and x_5. Employment shares are computed for 2010 from BLS tabulations.

SOC Code	Occupation Title	Employment Share
	x_1	
31	Healthcare Support Occupations	3.2%
35	Food Preparation and Serving Related Occupations	9.2%
37	Building and Grounds Cleaning and Maintenance	3.5%
39	Personal Care and Service Occupations	2.8%
	x_2	
41	Sales and Related Occupations	10.6%
43	Office and Administrative Support Occupations	17.7%
	x_3	
45	Farming, Fishing, and Forestry Occupations	0.3%
47	Construction and Extraction Occupations	4.2%
49	Installation, Maintenance, and Repair Occupations	4.0%
51	Production Occupations	6.5%
53	Transportation and Material Moving Occupations	7.0%
	x_4	
11	Management Occupations (Selected)	2.3%
17	Architecture and Engineering Occupations	1.7%
19	Life, Physical, and Social Science Occupations (Selected)	0.4%
27	Arts, Design, Entertainment, Sports, and Media Occupations	1.4%
29	Healthcare Practitioners and Technical Occupations	5.8%
33	Protective Service Occupations	2.6%
	x_5	
11	Management Occupations (Selected)	2.3%
13	Business and Financial Operations Occupations	3.0%
15	Computer and Mathematical Occupations	2.8%
19	Life, Physical, and Social Science Occupations (Selected)	0.4%
21	Community and Social Service Occupations	1.4%
23	Legal Occupations	0.8%
25	Education, Training, and Library Occupations	6.3%

The following table presents the Management Occupations (SOC-11) that are selected for group x_4:

SOC Code	Occupation Title
11102100	General and Operations Managers
11103100	Legislators
11302100	Computer and Information Systems Managers
11305100	Industrial Production Managers
11307102	Storage and Distribution Managers
11901101	Nursery and Greenhouse Managers
11901102	Crop and Livestock Managers
11901103	Aquacultural Managers
11901200	Farmers and Ranchers
11902100	Construction Managers
11903999	Education Administrators, All Other
11905100	Food Service Managers
11906100	Funeral Directors
11907100	Gaming Managers
11908100	Lodging Managers
11911101	Clinical Nurse Specialists
11919999	Managers, All Other
19406100	Social Science Research Assistants

The following table presents the Management Occupations (SOC-11) that are selected for group x_5:

SOC Code	Occupation Title
11101100	Chief Executives
11201100	Advertising and Promotions Managers
11202100	Marketing Managers
11202200	Sales Managers
11203100	Public Relations Managers
11301100	Administrative Services Managers
11303101	Treasurers and Controllers
11303102	Financial Managers, Branch or Department
11304000	Human Resources Managers
11304100	Compensation and Benefits Managers
11304200	Training and Development Managers
11306100	Purchasing Managers
11307101	Transportation Managers
11903100	Education Administrators, Preschool and Child Care Center/Program
11903200	Education Administrators, Elementary and Secondary School
11903300	Education Administrators, Postsecondary
11904100	Engineering Managers
11912100	Natural Sciences Managers
11913100	Postmasters and Mail Superintendents
11914100	Property, Real Estate, and Community Association Managers
11915100	Social and Community Service Managers
19406100	Social Science Research Assistants

The following table presents the Life, Physical, and Social Science Occupations (SOC-19) that are selected for group x_4:

SOC Code	Occupation Title
19101200	Food Scientists and Technologists
19101300	Soil and Plant Scientists
19102100	Biochemists and Biophysicists
19102200	Microbiologists
19102300	Zoologists and Wildlife Biologists
19102999	Biological Scientists, All Other
19103101	Soil and Water Conservationists
19103102	Range Managers
19103103	Park Naturalists
19103200	Foresters
19109999	Life Scientists, All Other
19203100	Chemists
19203200	Materials Scientists
19204100	Environmental Scientists and Specialists, Including Health
19204300	Hydrologists
19209999	Physical Scientists, All Other
19303999	Psychologists, All Other
19309102	Archeologists
19309999	Social Scientists and Related Workers, All Other
19401101	Agricultural Technicians
19401102	Food Science Technicians
19402100	Biological Technicians
19403100	Chemical Technicians
19404101	Geophysical Data Technicians
19404102	Geological Sample Test Technicians
19405101	Nuclear Equipment Operation Technicians
19405102	Nuclear Monitoring Technicians
19409100	Environmental Science and Protection Technicians, Including Health
19409200	Forensic Science Technicians
19409300	Forest and Conservation Technicians
19409999	Life, Physical, and Social Science Technicians, All Other

The following table presents the Life, Physical, and Social Science Occupations (SOC-19) that are selected for group x_5:

SOC Code	Occupation Title
19101100	Animal Scientists
19104100	Epidemiologists
19104200	Medical Scientists, Except Epidemiologists
19201100	Astronomers
19201200	Physicists
19202100	Atmospheric and Space Scientists
19204200	Geoscientists, Except Hydrologists and Geographers
19301100	Economists
19302100	Market Research Analysts
19302200	Survey Researchers
19303200	Industrial-Organizational Psychologists
19304100	Sociologists
19305100	Urban and Regional Planners
19309101	Anthropologists
19309200	Geographers
19309300	Historians
19309400	Political Scientists
19406100	Social Science Research Assistants

Appendix C

Appendix to Chapter 3

C.1 Additional Tables

Table C.1: Forbes Self-made Index

	Description	Fraction 2015
1	Inherited fortune but not working to increase it	7.00
2	Inherited fortune and has a role managing it	4.75
3	Inherited fortune and helping to increase it marginally	5.50
4	Inherited fortune and increasing it in a meaningful way	5.25
5	Inherited small or medium-size business and made it into a ten-digit fortune	8.50
6	Hired or hands-off investor who didn't create the business	2.25
7	Self-made who got a head start from wealthy parents and moneyed background	10.00
8	**Self-made who came from a middle- or upper-middle-class background**	**32.00**
9	**Self-made who came from a largely working-class background; rose from little to nothing**	**14.50**
10	**Self-made who not only grew up poor but also overcame significant obstacles**	**7.75**
	Our definition of "Self-made:" Groups 8 to 10	**54.25**

C.2 Misallocation in the Benchmark Economy

Our benchmark economy is distorted due to the existence of financial frictions in the form of borrowing constraints, and we can measure the effects of these distortions on aggregate TFP and output and compare them to those obtained in other studies. A large and growing literature frames the discussion on misallocation in terms of various wedges, such as capital, labor, and output wedges. The analysis in Hsieh and Klenow (2009) is particularly useful since, in a similar model environment, they study the degree of misallocation and its effect on TFP in manufacturing in China, India, and the United States. Hsieh and Klenow use detailed firm-level data from the U.S. Census of Manufacturers (1977, 1082, 1987, 1992, and 1997) and find that the TFP gains from removing all distortions (wedges), which equalizes the "Revenue Productivity" (TFPR) within each industry, is 36% in 1977, 31% in 1987, and 43% in 1997.

We will follow the approach in Hsieh and Klenow (2009) and will compute the same measures of misallocation for the U.S. as in their analysis. It is useful to briefly describe their approach as it applies to our framework. The final goods producer behaves competitively and uses an aggregated good, Q, and labor, L, in the production of the final good

$$Y = Q^{\alpha} L^{1-\alpha},$$

where Q aggregates the intermediate goods x_i in the following way

$$Q = \left(\int_i x_i^{\mu} di \right)^{1/\mu}.$$

Each intermediate-goods producer i produces a differentiated intermediate good using the production function $x_i = z_i k_i$, where z_i is the individual i's entrepreneurial ability and k_i is the amount of capital.

Instead of modeling and capturing the effect of a particular distortion, or distortions, the approach of Hsieh and Klenow, and the related misallocation literature, is to infer the underlying distortions and wedges in the economy by studying the extent to which the marginal revenue products of capital and labor differ across firms in the economy (or in a particular industry). This is based on the insight that absent any distortions, the marginal revenue products of capital and labor have to be equalized across all firms.[1]

[1] This is the case in the monopolistic competition models, such as in Hsieh and Klenow (2009). Alternatively, in environments such as in Lucas (1978) and Restuccia and Rogerson (2008), in which firms feature decreasing returns to scale, but produce the same homogeneous good, in the non-distorted economy the marginal products of capital and labor have to be equalized.

TFP in the Q sector. We will first focus on the Q-sector, the sector that produces the composite intermediate input Q by aggregating all the intermediate goods x_i. Under this alternative capital-wedge approach, the problem of each intermediate-goods producer is

$$\pi_i = \max_{k_i} p\left(z_i k_i\right) z_i k_i - \left(1+\tau_i^k\right)\left(R+\delta\right) k_i \ ,$$

where τ_i^k is a firm-specific capital wedge. The only input in the production function of the intermediate-goods producer is capital, and as a result only one wedge can be identified in the analysis. We choose to specify that wedge to be the capital wedge, but in principle it should be understood as capturing the effect of an output wedge.
The revenue TFP in sector Q for each firm i is

$$TFPR_{Q,i} \equiv \frac{p\left(x_i\right) x_i}{k_i} = \frac{1}{\mu}\left(1+\tau_i^k\right)\left(R+\delta\right).$$

The aggregate TFP in sector Q can be expressed as

$$TFP_Q = \left(\int_i \left(z_i \frac{\overline{TFPR_Q}}{TFPR_{Q,i}}\right)^{\frac{\mu}{1-\mu}} di\right)^{\frac{1-\mu}{\mu}} ,$$

where the average $TFPR_Q$ is given by

$$\overline{TFPR_Q} = \left(\int \frac{1}{TFPR_{Q,i}} \frac{p\left(x_i\right) x_i}{p_q Q} di\right)^{-1}.$$

In the non-distorted economy, without capital wedges, the level of TFP in the Q sector is

$$TFP_Q^* = \left(\int_i \left(z_i\right)^{\frac{\mu}{1-\mu}} di\right)^{\frac{1-\mu}{\mu}} \equiv \overline{z}.$$

Therefore, we can measure the improvement in TFP in the Q sector, Ω_Q, as a result of eliminating the capital wedges, or equivalently, as a result of eliminating the borrowing constraints:

$$\Omega_Q = \frac{TFP_Q^*}{TFP_Q} = \left(\int_i \left(\frac{\overline{z}}{z_i} \frac{TFPR_{Q,i}}{\overline{TFPR_Q}}\right)^{\frac{\mu}{1-\mu}} di\right)^{\frac{1-\mu}{\mu}} .$$

Table C.2 reports Ω_Q for various economies—the TFP in the Q sector in the non-distorted economy is 58% higher than in the benchmark economy, 51% higher than in the economy with a wealth tax, 54% higher than in the economy with consumption tax, 49% higher than in the economy with an optimal capital income tax, and 47% higher than in the economy

Table C.2: Hsieh and Klenow (2009) Efficiency Measure - Benchmark Model

	Benchmark	Tax Reform (τ_a)	Opt. Taxes (τ_k)	Opt. Taxes (τ_a)
TFP_Q	1.001	1.047	1.064	1.074
$\frac{TFP_Q^*}{TFP_Q}$	1.582	1.514	1.489	1.475
Mean TFPR	0.145	0.131	0.106	0.145
StD TFPR	0.054	0.048	0.039	0.053
p99.9	0.68	0.61	0.5	0.66
p99	0.35	0.32	0.27	0.35
p90	0.19	0.17	0.14	0.19
p50	0.14	0.12	0.1	0.14
p10	0.1	0.09	0.07	0.1

with an optimal wealth tax.

Wealth taxes give the higher TFP gains, allowing for better allocation of capital across firms, even without eliminating the borrowing constraints. The tax reform experiment to wealth taxes implies a TFP gain of 4.6% and optimal wealth taxes give a TFP gain of 7.3% with respect to our benchmark economy.

This can also be seen in the dispersion of TFPR of the different models. Recall that absent any constraints on the firms the TFPR would be equated across all of them, so there is higher misallocation in the economy the higher the dispersion of TFPR across firms. Table C.2 reports the standard deviation of TFPR and some of its percentiles.

Comparison with the Hsieh and Klenow (2009) results for the U.S.

In order to compare these results with the results reported in Hsieh and Klenow (2009) for the U.S., we need to note that the improvement in aggregate output, Ω_Y, as a result of eliminating the capital wedges in the economy can be expressed as

$$\Omega_Y = \frac{Y^*}{Y} = \left(\frac{TFP_Q^*}{TFP_Q}\right)^{\alpha} \left(\frac{K^*}{K}\right)^{\alpha} \left(\frac{L^*}{L}\right)^{1-\alpha}.$$

Since the model with capital wedges is static, the effect of the removal of the capital wedges on aggregate capital, K, and labor supply, L cannot be taken into account. The analysis in Hsieh and Klenow (2009), measures the improvement in total output as a result of an improvement in TFP in all industries. In our model, this corresponds to the improvement in TFP in the Q sector. Therefore, removing the capital wedges would increase total output, through its effect on TFP in the Q sector, by 20%.[2]

[2] Note that $\tilde{\Omega}_Y = \Omega_Q^{\alpha} = \Omega_Q^{0.40} = 1.20$.

Two things are important to point out. First, the magnitude of the misallocation in our benchmark economy is substantial, although a bit lower than the one measured in Hsieh and Klenow (2009) using micro data from manufacturing firms: 36% in 1977, 31% in 1987, and 43% in 1997. However it is in line with the level reported in ongoing research by Bils et al. (2017), who take into account measurement error in micro data, they find gains from removing distortions for the U.S. in the range of 20%. In any case, it is worth noting several differences between our framework and that of Hsieh and Klenow (2009). Our benchmark economy is parametrized based on moments from the entire economy, not just the manufacturing sector. Second, our benchmark model is a dynamic model and any changes in the financial frictions will affect aggregate capital accumulation and aggregate labor supply. The misallocation calculations above do not take those changes into account. It is clear, however, that eliminating the financial friction would increase the aggregate capital stock K and lead a larger increases in total output than measured above. The effect on aggregate labor supply is less obvious.